Generalizing Unweighted Network Measures Using the Effective Cardinality*

Sherief Abdallah
British University in Dubai
University of Edinburgh (Honorary Fellow)
sherief.abdallah@ieee.org

ABSTRACT

Several important complex network measures that helped discovering common patterns across real-world networks ignore edge weights, an important information in real-world networks. We propose a new methodology for generalizing measures of unweighted networks through a generalization of the cardinality concept of a set of weights. The key observation here is that many measures of unweighted networks use the cardinality (the size) of some subset of edges in their computation. For example, the node degree is the number of edges incident to a node. We define the *effective cardinality*, a new metric that quantifies how many edges are effectively being used, assuming that an edge's weight reflects the amount of interaction across that edge.

We prove that a generalized measure, using our proposed effective cardinality metric, reduces to the original unweighted measure if there is no disparity between weights. This property ensures that laws that govern the original unweighted measure will also govern the generalized measure when weights are equal. We also prove that our generalization ensures a partial ordering among sets of weighted edges that is consistent with the original unweighted measure, unlike previously developed generalizations. We illustrate the applicability of our method by generalizing four unweighted network measures: the node degree, the clustering coefficient, the dyadicity, and the heterophilicity. As a case study, we analyze four real-world, weighted, social networks using one of our generalized measures: the C-degree. The analysis shows that the distribution of the generalized degree follows a similar pattern to the traditional degree but with steeper decline. There is also a common pattern governing the ratio between the generalized degree and the traditional degree.

1. INTRODUCTION

Several important complex network measures that helped discovering patterns common in real-world networks [17, 3, 16] ignore edge weights, an important information in real-world networks. We propose here a new methodology for generalizing measures of unweighted networks through a generalization of the cardinality concept of a set of weighted edges. The key observation here is that many measures of unweighted networks use the cardinality (the size) of some subset of edges in their computation. For example, the node degree is the number of edges incident to a node. The clustering coefficient of a node is the ratio between the number of edges between its neighbors and the number of all possible edges among the neighbors.[1] We propose here the *effective cardinality* metric, which quantifies how many edges are effectively being used among a set of weighted edges. By simply replacing the traditional cardinality with the effective cardinality, one can generalize unweighted network measures to take weights into account. The central assumption here is that an edge's weight reflects the amount of interaction across that edge.

A generalization of unweighted network measures that still upholds the properties of unweighted network measures (as we prove regarding our generalization) is significant, because it bridges the gap between the extensive research made using the unweighted network measures and the research on weighted networks. Furthermore, it allows more accurate analysis of the networks that were previously analyzed using unweighted network measures. For example, it is known that the degree distributions of several real world networks are consistent with the power law [7]. However, if one takes the disparity of interactions into account, will the *effective* degree distributions of these networks be consistent with the power law?

There have been several attempts to generalize the measures of unweighted networks (or at least some of them) to account for weights [4, 5, 1, 15]. Most of the previous work focused on generalizing individual measures, such as the clustering coefficients [4, 15]. Perhaps the most related work in generalizing unweighted network measures is the ensemble approach [1]. The first step in the ensemble approach was randomly generating a collection of unweighted networks from the original weighted network (an edge was generated proportional to its weight). The desired (unweighted) measure was computed for each network in the ensemble and then averaged to produce the generalized measure.

We provide here a generalization methodology that applies to a large number of unweighted measures. More importantly, unlike the previous attempts, we prove that our generalized measures become identical to the original unweighted measures if there is no disparity between weights (i.e. all weights are equal). This property ensures that all the results that applies for an unweighted measure directly follows for our generalization of the same measure if the weights are equal. We also prove that the effective cardinality, the heart of our generalization, imposes a partial ordering among

*A pre-print, up-to-date, longer version of the paper is available through arXiv, arXiv:0905.4169v1

[1]Other examples include heterophilicity and dyadicity. We describe these measures in further detail later.

The 3rd SNA-KDD Workshop (SNA-KDD'09), June 28, 2009, Paris, France

sets of weighted edges that is consistent with the traditional cardinality. Intuitively this means that the smaller the disparity between weights, the closer the generalized measure is to the unweighted measure. This point will become clearer in Section 3 where we discuss in detail the different properties of the effective cardinality.

We illustrate the applicability of our method by generalizing four unweighted network measures that has been used in the literature to discover interesting network patterns: the node degree, the clustering coefficient, the dyadicity, and the heterophilicity. Furthermore, as a case study, we analyze four real world weighted networks using our generalization of the degree measure and show that the power-law still holds for the generalized degree but with steeper decline than the traditional degree distribution. We also report an interesting pattern that govern the ratio between the generalized degree and the traditional degree.

The paper is organized as follows. Section 2 describes our proposed effective cardinality metric. Section 3 discusses and proves interesting properties of our proposed metric. Then in section 4 we describe some unweighted network measures and show how they can be generalized using the effective cardinality. In Section 5 we discuss in more detail our generalization of the degree measure and analyze four real world weighted networks. We conclude in Section 6.

2. THE EFFECTIVE CARDINALITY

Let $E' = \{e_1, ..., e_n\} \subseteq E$ be the subset of edges used by a particular network measure, where $n = |E'|$ is the cardinality of this subset and E is the set of all network edges. Let $w(e)$ be the weight corresponding to edge $e \in E$. We assume $\forall e \in E : w(e) \geq \epsilon > 0$. The heart of our methodology is a generalized definition of the cardinality of edges that takes weights into account, which we call the *effective cardinality*:

$$c(E') = \begin{cases} 0 & \text{if } E' \text{ is empty} \\ 2^{\left(\sum_{e\in E'} \frac{w(e)}{\sum_{o\in E'} w(o)} \log_2 \frac{\sum_{o\in E'} w(o)}{w(e)}\right)} & otherwise \end{cases}$$

The effective cardinality, by definition, is a real number (not discrete as the traditional cardinality). Intuitively, the quantity $\frac{w(e)}{\sum_{o\in E'} w(o)}$ represents the probability of an interaction over an edge e among all the edges in E'. The set $\left\{\frac{w(e)}{\sum_{o\in E'} w(o)} : e \in E'\right\}$ is a probability distribution and the quantity $\sum_{e\in E'} \left[\frac{w(e)}{\sum_{o\in E'} w(o)} \log_2 \frac{\sum_{o\in E'} w(o)}{w(e)}\right]$ is the entropy of this probability distribution, which measures the disparity between the weights: the more uniform the weights are, the higher the entropy and vice versa. The purpose of the power 2 is to convert the entropy back to the number of edges that are effectively being used. The following section discusses and proves the distinguishing characteristics of the effective cardinality metric.

3. PROPERTIES OF THE EFFECTIVE CARDINALITY

The effective cardinality satisfies three intuitive properties (proofs are given shortly after):

1. **Preserving maximum cardinality:** $\forall E' : c(E') \leq |E'|$. Furthermore, $c(E') = |E'|$ iff $\forall e \in E' : w(e) = C$, where C is some constant. In other words, the effective cardinality is maximum when all weights are equal.

2. **Preserving minimum cardinality:** $c(E') = 0$ iff E' is an empty set. Furthermore, $c(E')$ is close to 1 iff $\exists u \in E' : w(u) >> \epsilon$ and $\forall v \neq u : w(v) = \epsilon$. In other words, the effective cardinality is almost one when all edges, except one edge, have almost zero weights.

3. **Consistent partial order over weighted sets:** any function that maps a set of real numbers (weights) to a single real number imposes an implicit partial order. The effective cardinality imposes, arguably, the simplest partial order that is consistent with the above two properties. If the two sets of weighted edges have the same size, the same strength, and their individual weights are the same except for two edges, then set with more uniform weights has higher effective cardinality. More formal definition of the property is given in the proof section.

The intuition of the three properties can be clarified through the following example. Suppose there are four sets of edges with corresponding sets of weights $W_1 = \{5, 5, 5, 5\}, W_2 = \{9, 5, 5, 1\}, W_3 = \{9, 8, 2, 1\}$ and $W_4 = \{20-3\epsilon, \epsilon, \epsilon, \epsilon\}$. The above properties then require the effective cardinality measure to impose the following ordering: $|W_1| = c(W_1) > c(W_2) > c(W_3) > c(W_4) \approx 1$. We prove each of these properties in the remainder of this section. Note that the ensemble approach [1] made no guarantees with regard to the partial order over sets of weighted edges. For example, the two set of weights $W_2 = \{9, 5, 5, 1\}$ and $W_3 = \{9, 8, 2, 1\}$ will have the same generalized degree under the ensemble method. Using the effective cardinality, the generalized degree (described in detail in Section 5) of W_2 is strictly higher than the generalized degree of W_3.

LEMMA 1. *The effective cardinality satisfies the maximum cardinality property.*

PROOF. When all the weights are equal to a constant C we have

$$\forall e \in E' : \frac{w(e)}{\sum_{o\in E'} w(o)} = \frac{C}{C|E'|} = \frac{1}{|E'|}$$

We then have

$$\begin{aligned} c(E') &= 2^{\sum_{e\in E'} \frac{1}{|E'|} \log_2(|E'|)} \\ &= 2^{\log_2(|E'|)} \\ &= |E'| \end{aligned}$$

In other words, both the cardinality and the effective cardinality of a weighted set of edges become equivalent when the weights are uniform. The effective cardinality is also maximum in this case, because the exponent is the entropy of the weight probability distribution (which is maximum when weights are uniform over edges). □

LEMMA 2. *The effective cardinality satisfies the minimum cardinality property.*

PROOF. When the set of edges is empty, then the effective cardinality is zero by definition. When all weights are close to zero except only one weight that is much bigger than zero, then weight probability distribution is almost deterministic and the entropy is close to zero, therefore the effective cardinality will be close to 1. □

LEMMA 3. *The effective cardinality satisfies the consistent partial order property.*

PROOF. Let E'_1 and E'_2 be two (edge) sets such that $|E'_1| = |E'_2|$ (both have the same cardinality). Let W_1 and W_2 be the corresponding sets of weights, where $\sum_{e1\in E'_1} w(e1) = \sum_{e2\in E'_2} w(e2) =$

S (the total weights are equal). Furthermore, let $|W_1 \bigcap W_2| = n-2, \{w_{11}, w_{12}\} = W_1 - W_2, \{w_{21}, w_{22}\} = W_2 - W_1$, where the '$-$' operator is the "set difference" operator (the two sets share the same weights except for two elements in each set), and $|w_{11} - w_{12}| < |w_{21} - w_{22}|$ (the weights of W_1 are more uniform than the weights of W_2). Without loss of generality, we can assume that $w_{11} \geq w_{12}$ and $w_{21} \geq w_{22}$, therefore $w_{11} - w_{12} < w_{21} - w_{22}$. We then have

$$\frac{w_{11} + w_{12}}{S} = 1 - \sum_{w \in W_1 \bigcap W_2} \frac{w}{S} = \frac{w_{21} + w_{22}}{S} = L$$

, therefore

$$L \geq \frac{w_{21}}{S} > \frac{w_{11}}{S} \geq \frac{L}{2} \geq L - \frac{w_{11}}{S} > L - \frac{w_{21}}{S}$$

where $\frac{w_{12}}{S} = L - \frac{w_{11}}{S}$ and $\frac{w_{22}}{S} = L - \frac{w_{21}}{S}$. Then from Lemma 4 we have $h(L, \frac{w_{11}}{S}) > h(L, \frac{w_{21}}{S})$, or

$$-\frac{w_{11}}{S} \lg(\frac{w_{11}}{S}) - (L - \frac{w_{11}}{S}) \lg(c - \frac{w_{11}}{S}) >$$
$$-\frac{w_{21}}{S} \lg(\frac{w_{21}}{S}) - (L - \frac{w_{21}}{S}) \lg(c - \frac{w_{21}}{S})$$

Therefore $H(E_1) > H(E_2)$, because the rest of the entropy terms (corresponding to $W_1 \bigcap W_2$) are equal, and consequently $c(E_1) > c(E_2)$. $\square$

LEMMA 4. *The quantity $h(C, x) = -x \lg(x) - (C-x) \lg(C-x)$ is symmetric around and maximized at $x = \frac{C}{2}$ for $C \geq x > 0$.*

PROOF. Symmetry around $c/2$: $h(C, \frac{C}{2}+\delta) = -(\frac{C}{2}+\delta) \lg(\frac{C}{2}+\delta) - (\frac{C}{2}-\delta) \lg(\frac{C}{2}-\delta) = h(C, \frac{C}{2}-\delta)$. Maximum at $c/2$: $h(C,x)$ is maximized when $\frac{\partial h(C,x)}{\partial x} = 0 = -1 - \lg x + 1 + \lg(C-x)$, therefore $x = C - x = \frac{C}{2}$. $\square$

4. GENERALIZING UNWEIGHTED NETWORK MEASURES

In principal, any unweighted network measure which uses the cardinality of some subset of edges can be generalized using the effective cardinality. Furthermore, the resulting generalized measures will inherit the three properties of the effective cardinality. We here present four example generalizations of unweighted network measures: the degree, the clustering coefficient, the dyadicity, and the heterophilicity.

A node's degree is the number of edges incident to the node, or $|E_i|$, where E_i is the set of edges incident to node i. Using the effective cardinality metric, a generalization of the degree is simply $c(E_i)$. The following section presents a detailed analysis of four real world networks using the generalized degree and discusses its relationship to the traditional degree.

The clustering coefficient is a measure that quantifies the clustering or connectivity among a node's neighbors. When averaged over all nodes, the clustering coefficient represents the connectivity of the whole network. The clustering coefficient was an important property for identifying small world networks [17] and is given by the equation $\frac{|E_{N(i)}|}{CLIQUE(|N(i)|)}$, where $N(i)$ is the set of neighboring nodes to node i, $E_{N(i)}$ is the set of edges between the nodes in $N(i)$, and the function $CLIQUE$ returns the number of edges in a clique of size $|N(i)|$). The generalized clustering coefficient of a node i using the effective cardinality is simply $\frac{c(E_{N(i)})}{CLIQUE(|N(i)|)}$.

Two recent measures were used to study the correlation between the types of nodes (node classes) in a network and the network structure: the dyadicity and heterophilicity [16]. The dyadicity of a graph equals $\frac{c(E_{within})}{n_{within}}$, where E_{within} is the set of edges within a set of nodes of the same type (a class of nodes) and n_{within} is the expected number of edges within the same class of nodes if there was no correlation between the node class and the network structure. Intuitively, the dyadicity measure quantifies the strength of connections between nodes of the same type and whether it is above average.[2] The heterophilicity of a graph equals $\frac{c(E_{across})}{n_{across}}$, where E_{across} is the set of edges across two classes of nodes and n_{across} is the expected number of edges across the two classes if there was no correlation between the node class and the network structure. The dyadicity can be generalized, using the effective cardinality, to be $\frac{c(E_{within})}{n_{within}}$ and similarly the heterophilicity can be generalized to be $\frac{c(E_{across})}{n_{across}}$.

The following section discusses in detail our generalization of the degree measure, based on the effective cardinality, and analyzes four real world networks using this generalization.

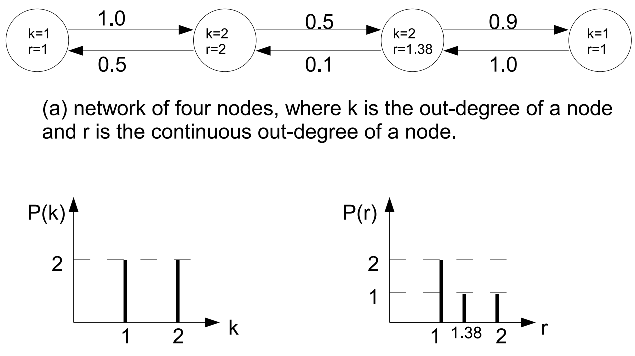
5. CASE STUDY: THE CONTINUOUS DEGREE, A GENERALIZATION OF THE DEGREE MEASURE

A key measurement that has been used extensively in analyzing networks is the degree of a node. The degree distribution is a common method for summarizing the degrees of all network nodes into one measure that characterizes complex networks [3, 8, 7]. Implicitly, the degree measure assumes uniform interaction across each node's neighbors, similar to other measures of unweighted networks that ignore any disparity in the weights. This can result in giving an incorrect perception of the *effective* node degree. For example, a person may have 10 or more acquaintances but mainly interacts with only two of them (friends). Should that person be considered 2 times more connected than a person with only 5 acquaintances but also interacting primarily with two of them?

Most of the previous work that used the degree measure defined some cutoff threshold in order to either include or exclude a weighted edge and then computed the degree distribution normally [6, 9]. Such an approach, however, does not properly handle the disparity of interaction among neighbors, but rather approximates a weighted network with an unweighted network.

Surveying all network measures that were proposed to analyze weighted networks and had some similarity to the degree measure is beyond the scope of this paper. Instead, we focus on a sample of these measures that are mostly related to our contribution (interested reader may refer to survey papers on the subject such as [5]). The *weight distribution* $P(w)$ is similar to the degree distribution except that it measures the frequency of a particular edge weight. The *strength* of a node is the summation of all weights incident to a node and it becomes identical to the node's degree if all weights are equal to 1. A more recent work [10] analyzed a graph's total weight, $\sum_{e \in E} w(e)$, against the graph's total number of edges, $|E|$, over time. That work also analyzed the degree of a node, $k(v)$, against the node's strength, $s(v)$. While useful, none of these measures captured the disparity in interaction between a node and its neighbors. The network measure $Y(v) = \sum_{e \in E(v)} \left(\frac{w(e)}{s(v)}\right)^2$ successfully captured the disparity of interaction within a node v [2]. However, the Y measure is not a *generalization* of the degree measure as it fails to satisfy the first two properties we define in Section 3. In other words, if the weights are equal, the Y measure of a node

[2]There are other network measures that also quantified the strength of connections within a class (community) of nodes, such as the modularity measure [12].



(a) network of four nodes, where k is the out-degree of a node and r is the continuous out-degree of a node.

(b)The degree distribution of the network in (a)

(c)The continuous degree distribution of the network in (a)

Figure 1: Continuous vs discrete degree distributions.

does not become equal to the node's degree.

Using our definition of effective cardinality, a generalization of the degree measure, which we call the continuous degree or the C-degree, is given by the following equation:

DEFINITION 5. *The* C-degree *of a node i in a network is $r(i)$, where*

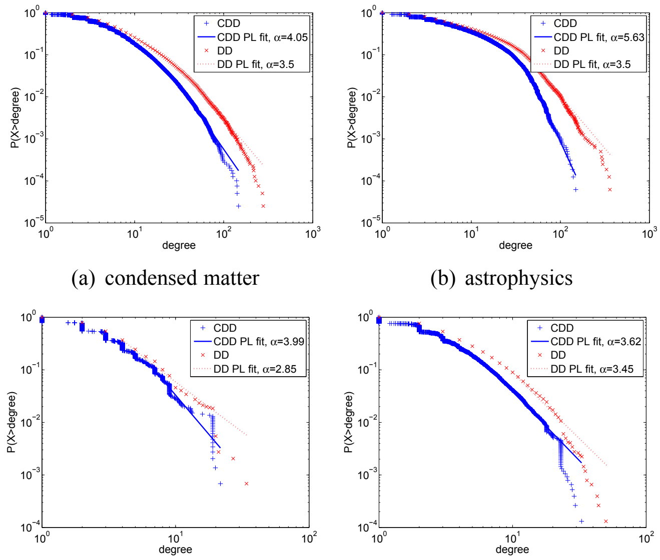
$$r(i) = c(E_i) = \begin{cases} 0 & \text{if } i \text{ is disconnected} \\ 2^{\left(\sum_{e \in E_i} \frac{w(e)}{s(i)} \log_2 \frac{s(i)}{w(e)}\right)} & \textit{otherwise} \end{cases}$$

Where E_i is the set of edges incident to node i and $s(i)$ is the strength of node i. Figure 1 compares the continuous degree distribution to the (discrete) degree distribution in a simple weighted network of four nodes. A node on the boundary has an out degree of 1, while an internal node has an out degree of 2. Intuitively, however, only one of the internal nodes is fully utilizing its degree of 2 (the one to the left), while the other node (to the right) is mostly using one neighbor only. The C-degree measure captures this and shows that only one internal node has a C-degree of 2 while the other internal node has a C-degree of 1.38.

The C-degree inherits the three properties we described earlier with respect to the traditional node degree. The C-degree of a node is maximum and equals the traditional discrete degree when all the weights incident to the node are equal. The C-degree of a connected node is minimum and very close to one if all edges incident to the node have weights that are almost zero except one edge that has a weight much larger than zero. And finally, everything else being equal, a node with more uniform weights incident to it has higher C-degree than a node with less uniform weights incident to it. As mentioned earlier, the three properties ensure that the four sets of weights $W(v1) = \{5, 5, 5, 5\}, W(v2) = \{9, 5, 5, 1\}, W(v3) = \{9, 8, 2, 1\}$ and $W(v4) = \{20 - 3\epsilon, \epsilon, \epsilon, \epsilon\}$ will have corresponding C-degree respecting the following inequality $k(v1) = r(v1) > r(v2) > r(v3) > r(v4) \approx 1$.

We have analyzed four real world weighted networks [3] that capture coauthorships between scientists. Three of which were extracted from preprints on the E-Print Archive [13]: condensed matter (updated version of the original dataset that includes data between Jan 1, 1995 and March 31, 2005), astrophysics, and high-energy theory. The fourth network represents coauthorship of scientists in network theory and experiment [14]. The weight between two scientists i and j reflects the strength of their collaboration and is given by the equation $w_{ij} = \sum_k \frac{\delta_i^k \delta_j^k}{n_k - 1}$, where $\delta_i^k = 1$ if scientist i was a co-author of paper k and n_k is the number of co-authors for paper k[11].

It was shown that the degree distribution of several real networks is consistent with the power law [5, 12]. A degree distribution follows the power law if $P(k) \propto k^{-\alpha}$, where α is a constant. Figure 2 displays the C-degree distribution (CDD) and the (discrete) degree distribution (DD) for the four collaboration network. The figure uses log-log scale with the power law fit (based on [7][4]). The CDD preserves a behavior similar to the traditional degree, but with steeper decline, which is consistent with Lemma 1.



(a) condensed matter

(b) astrophysics

(c) network theory

(d) high-energy theory

Figure 2: Comparing the discrete degree distribution (DD) with the continuous degree distribution (CDD) for the four collaboration networks. The power law fit (PL fit) is also shown with the associated power.

One would expect that as the degree of a node increases, the node will interact primarily with a smaller subset of neighbors, particularly in social networks where humans have limited communication capacity. To verify this intuition, we define the *degree utilization* metric as the ratio between the C-degree and the degree of a node: $u(v) = \frac{r(v)}{k(v)}$. The degree utilization metric captures the percentage of links that a node uses effectively, therefore we expect the degree utilization to decrease as the degree increases. Figure 3 plots the degree utilization against the (discrete) degree for the four collaboration networks. A common pattern emerges in the four networks. For low degrees, the degree utilization is relatively high (a node with few links makes the best of them). For node degree greater than some constant the bias towards high degree utilization disappears. However, and to our surprise, a *cone* is observed, which starts wide at low degrees and gets narrower as the degree increases (the average degree utilization is plotted as a line in the figure). In other words, for degrees above some threshold, nodes vary in their utilization of their available links. However, this variation reduces as the degree increases, while the mean remains relatively stable.

[3]Available through http://www-personal.umich.edu/ mejn/netdata/

[4]Source code available from http://www.santafe.edu/ãaronc/powerlaws/

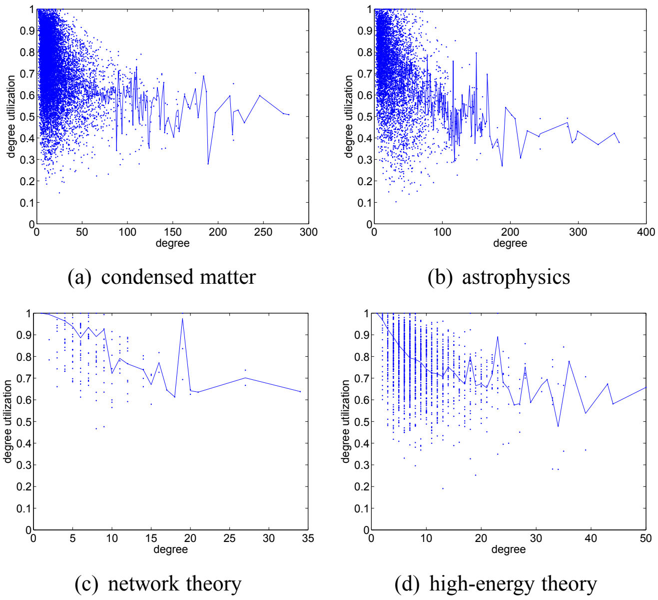


Figure 3: Scatter plot of a node degree against its degree utilization for the four collaboration networks. the average utilization per degree is also plotted.

6. CONCLUSION

We proposed a new methodology for generalizing measures of unweighted networks by defining the *effective cardinality*, a new metric which quantifies how many edges, of a particular subset of edges, are effectively being used. We illustrated the applicability of our method by generalizing four unweighted network measures: the node degree, the clustering coefficient, the dyadicity, and the heterophilicity. Furthermore, we compared the generalized degree to the traditional degree using four real world networks and showed that the power-law still holds for the generalized degree but with steeper decline. We also investigated the ratio between the generalized degree and the traditional degree and showed that on average the ratio is lower bounded, even for nodes with high-degree.

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