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Existence of resolvable group divisible designs with block size four I

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Abstract

It is proved in this paper that for $m \not\equiv 0, 2, 6, 10 \pmod{12}$ there exists a resolvable group divisible design of order v, block size 4 and group size m if and only $v \equiv 0 \pmod 4$, $v \equiv 0 \pmod m$, $v - m \equiv 0 \pmod 3$, except when $(3, 12)$ and except possibly when $(3, 264)$, $(3, 372)$, $(8, 80)$, $(8, 104)$, $(9, 396)$ $(40, 400)$ or $(40, 520)$.

Keywords: Resolvable group divisible design; Labeled resolvable design; Kirkman frame

1. Introduction

Let v be a positive integer, let K and M be two sets of positive integers. A group divisible design (GDD), denoted $\mathrm{GD}(K, M; v)$, is a triple $(X, \mathbf{G}, \mathbf{A})$, where X is a v-set, $\mathbf{G}$ is a set of subsets (called groups) of X, $\mathbf{G}$ partitions X, and $\mathbf{A}$ is a set of subsets (called blocks) of X such that

(1) $|G| \in M$ for each $G \in \mathbf{G}$,
(2) $|B| \in K$ for each $B \in \mathbf{A}$,
(3) $|B \cap G| \leqslant 1$ for each $B \in \mathbf{A}$ and each $G \in \mathbf{G}$,
(4) Each pair of elements of X from distinct groups is contained in a unique block.

Let $(X, \mathbf{G}, \mathbf{A})$ be a $\mathrm{GD}(K, M; v)$, the group-type, or type, is the multiset $\{|G|: G \in \mathbf{G}\}$. We usually use an "exponential" notation to denote the group-type: $(X, \mathbf{G}, \mathbf{A})$ is called

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a K-GDD of type T, where

$$T = \prod_{1 \leqslant i \leqslant s} g_i^{u_i}$$

if $\mathbf{G}$ contains u_i groups of size g_i for $1 \leqslant i \leqslant s$, and $\sum_{1 \leqslant i \leqslant s} g_i u_i = v$. If $K = \{k\}$ and $M = \{m\}$, then the $\mathrm{GD}(K, M, v)$ is called uniform and simply denoted $\mathrm{GD}(k, m; v)$.

In a $\mathrm{GD}(K, M, v)$ $(X, \mathbf{G}, \mathbf{A})$, a parallel class is a set of blocks which partitions $\mathbf{X}$. If $\mathbf{A}$ can be partitioned into parallel classes, the $\mathrm{GD}(K, M; v)$ is called resolvable and denoted $\mathrm{RGD}(K, M; v)$.

The existence of resolvable uniform group divisible designs has been studied extensively. An $\mathrm{RGD}(k, 1; v)$ is known as a Kirkman system and denoted $\mathrm{KS}(2, k, v)$, An $\mathrm{RGD}(k, k-1; v)$ is called a nearly Kirkman system and denoted $\mathrm{NKS}(2, k, v)$. An $\mathrm{RGD}(k, m; mk)$ is called a resolvable transversal design and denoted $\mathrm{RTD}(k, m)$. It is well known that the existence of an $\mathrm{RTD}(k, m)$ is equivalent to the existence of $k-1$ mutually orthogonal Latin squares of order m.

Since there are precisely $r = (v-m)/(k-1)$ parallel classes in an $\mathrm{RGD}(k, m; v)$, then it can be easily seen that the following conditions are necessary for the existence of an $\mathrm{RGD}(k, m; v)$:

$$\begin{aligned} &v \equiv 0 \pmod{m},\\ &v \equiv 0 \pmod{k},\\ &v - m \equiv 0 \pmod{(k-1)}. \end{aligned} \tag{1}$$

In the case $k = 3$, it is proved that necessary conditions (1) for the existence of an $\mathrm{RGD}(3, m; v)$ are also sufficient with three exceptions.

Theorem 1.1 (Assaf and Hartman [1], Rees and Stinson [5], Rees [7]). *There exists an* $\mathrm{RGD}(3, m; v)$ *if and only if*

$$v \equiv 0 \pmod{3}, \quad v - m \equiv 0 \pmod{2},$$

except when $(v, m) = (6, 2)$, $(12, 2)$ *and* $(18, 6)$.

In this paper, we investigate the existence of resolvable uniform group divisible designs with block size 4. In this case, necessary conditions (1) can be stated in the following form.

Lemma 1.1. *If there exists an* $\mathrm{RGD}(4, m; v)$, *then* $v \geqslant 4m$ *and*

(i) $v \equiv 4m \pmod{12m}$ *if* $m \equiv 1, 5, 7, 11 \pmod{12}$,
(ii) $v \equiv 4m \pmod{6m}$ *if* $m \equiv 2, 10 \pmod{12}$,
(iii) $v \equiv 0 \pmod{4m}$ *if* $m \equiv 3, 9 \pmod{12}$,
(iv) $v \equiv m \pmod{3m}$ *if* $m \equiv 4, 8 \pmod{12}$,

(v) $v \equiv 0 \pmod{2m}$ *if* $m \equiv 6 \pmod{12}$,
(vi) $v \equiv 0 \pmod{m}$ *if* $m \equiv 0 \pmod{12}$.

For $m = 1$, a complete solution to the existence of RGD$(4, 1; v)$ is obtained.

Lemma 1.2 (Hanani [4]). *There exists an* RGD$(4, 1; v)$, *i.e. a Kirkman system* KS$(2, 4; v)$, *if and only if* $v \equiv 4 \pmod{12}$.

For $m = 3$, the following result is obtained.

Lemma 1.3 (Shen [10]). *Let*

$$E = \{84, 120, 132, 180, 216, 264, 312, 324, 372, 456, 552, 648, 660, 804, 852, 888\}.$$

If $v \notin E$, *then there exists an* RGD$(4, 3; v)$, *if and only if*

$$v \equiv 0 \pmod{12}, \quad v \geqslant 24. \tag{2}$$

The existence of an RGD$(4, 3; v)$ for $v \in \{120, 180, 216, 312, 324, 648, 888\}$ is proved in [6]. An RGD$(4, 3; 84)$ is constructed in [11]. The existence of an RGD$(4, 3; v)$ for $v \in \{132, 456, 552, 660, 804, 852\}$ is proved in [12]. Thus, we have the following almost complete solution for the existence of nearly Kirkman systems with block size 4.

Lemma 1.4. *There exists an* RGD$(4, 3; v)$ *if and only if*

$$v \equiv 0 \pmod{12}, \quad v \geqslant 24$$

with the possible exceptions of $v = 264, 372$.

The purpose of this paper is to give an almost complete solution for the existence of resolvable group divisible designs with block size 4 and group size m where $m \equiv 1, 3, 4, 5, 7, 8, 9, 11 \pmod{12}$.

2. Labeled resolvable designs

To provide powerful direct constructions for resolvable group divisible designs, we need the concept of labeled resolvable block designs.

A λ-fold balanced incomplete block design of order v and block size k, denoted $B(k, \lambda; v)$, is a pair $(X, \mathbf{B})$ where X is a v-set and $\mathbf{B}$ is a collection of k-subsets (called blocks) of $\mathbf{V}$ such that each 2-subset of X is contained in precisely λ blocks. A $B(k, \lambda; v)$ is called resolvable and denoted RB$(k, \lambda; v)$ if all the blocks can be partitioned into parallel classes.

Let $(X, \mathbf{B})$ be a $B(k, \lambda; v)$ where $X = \{a_1, a_2, \ldots, a_v\}$ is a totally ordered v-set with ordering $a_1 < a_2 < \cdots < a_v$. For each block $B = \{x_1, x_2, \ldots, x_k\}$, we suppose that $x_1 < x_2 < \cdots < x_k$.

Let

$$\varphi : \mathbf{B} \to Z_\lambda^{\binom{k}{2}}$$

be a mapping where for each $B = \{x_1, x_2, \ldots, x_k\} \in \mathbf{B}$,

$$\varphi(B) = (\varphi(x_1,x_2), \ldots, \varphi(x_1,x_k), \varphi(x_2,x_3), \ldots, \varphi(x_{k-1},x_k)), \quad \varphi(x_i,x_j) \in Z_\lambda$$
$$\forall 1 \leqslant i < j \leqslant k.$$

For convenience, for $\{x, y\} \subset$X with $y < x$, we define $\varphi(x, y)$ to be

$$\varphi(x, y) \equiv -\varphi(y, x) \pmod{\lambda}.$$

If there is a mapping φ satisfying the following two conditions:

(i) For each pair $\{x, y\} \subset X$ with $x < y$, let $B_1, B_2, \ldots, B_\lambda$ be the λ blocks containing $\{x, y\}$ and let $\varphi(x, y)_i$ be the values of $\varphi(x, y)$ corresponding to B_i, $1 \leqslant i \leqslant \lambda$. Then for $1 \leqslant i, j \leqslant \lambda$,

$$\varphi(x, y)_i \equiv \varphi(x, y)_j \pmod{\lambda}$$

if and only if $i = j$.

(ii) For each block $B = \{x_1, x_2, \ldots, x_k\}$, we have

$$\varphi(x_r, x_s) + \varphi(x_s, x_t) \equiv \varphi(x_r, x_t) \pmod{\lambda} \quad \forall 1 \leqslant r < s < t \leqslant k.$$

Then the $B(k, \lambda; v)$ is called a labeled block design and denoted LB$(k, \lambda; v)$, its blocks will be denoted in the following form:

$$(x_1, x_2, \ldots, x_k; \varphi(x_1,x_2), \ldots, \varphi(x_1,x_k), \varphi(x_2,x_3), \ldots, \varphi(x_{k-1},x_k)).$$

A labeled RB$(k, \lambda; v)$ is denoted LRB$(k, \lambda; v)$.

As examples, we construct an LRB$(4, 3; v)$ for $v = 8, 12$, which can be found in [10].

Example 2.1. An LRB$(4, 3; 8)$.
$X = Z_7 \cup \{\infty\}$ with ordering: $0 < 1 < 2 < \cdots < 6 < \infty$. Parallel classes:

$$\begin{array}{lll} P_i: & (\ i,\ \ i+1,\ i+3,\ i+6;\ 0,0,1,0,1,1\ ), & \\ & (\ i+2,\ i+4,\ i+5,\ \infty;\ 1,2,0,1,2,1\ ), & i \in Z_7. \end{array}$$

Example 2.2. An LRB$(4, 3; 12)$.
$X = Z_{11} \cup \{\infty\}$ with ordering: $0 < 1 < 2 < \cdots < 10 < \infty$. Parallel classes:

$$\begin{array}{lll} P_i: & (\ i,\ \ i+1,\ i+4,\ i+10;\ 2,2,0,0,1,1\ ), & \\ & (\ i+2,\ i+5,\ i+6,\ i+8;\ 2,0,0,1,1,0\ ), & \\ & (\ i+3,\ i+7,\ i+9,\ \infty;\ 1,2,1,1,0,2\ ), & i \in Z_{11}. \end{array}$$

The following LRB$(4, 3; 16)$ can be found in [12].

Example 2.3. An LRB(4, 3; 16).
$X = Z_{15} \cup \{\infty\}$ with ordering: $0 < 1 < 2 < \cdots < 14 < \infty$. Parallel classes:

$$\begin{array}{llllll}
P_i: & (\; i, & i+1, & i+2, & i+5; & 0,1,0,1,0,2 \;), \\
& (\; i+8, & i+9, & i+12, & i+3; & 2,2,1,0,2,2 \;), \\
& (\; i+4, & i+6, & i+11, & i+13; & 0,1,0,1,0,2 \;), \\
& (\; i+7, & i+10, & i+14, & \infty; & 1,2,0,1,2,1 \;), \quad i \in Z_{15}.
\end{array}$$

For the application of labeled resolvable block designs in the construction of resolvable group divisible designs, we have the following theorem.

Theorem 2.1 (Shen [9]). *If there is an* LRB$(k, \lambda; v)$ *with* $\lambda = m$, *then there exists an* RGD$(k, m; mv)$.

Thus, by Theorem 2.1, the existence of an RGD$(4, 3; v)$ for $v \in \{24, 36, 48\}$ follows from Examples 2.1–2.3.

Lemma 2.1. *There exists an* LRB$(4, 9; v)$ *for* $v \in \{8, 12\}$.

Proof. Let $X = Z_{v-1} \cup \{\infty\}$ with ordering $0 < 1 < 2 < \cdots < v-2 < \infty$.
(i) $v = 8$, parallel classes: P_{1i}, P_{2i}, P_{3i}, $i \in Z_7$:

$$\begin{array}{llllll}
P_{1i}: & (\; i, & i+1, & i+2, & i+4; & 1,1,1,0,0,0 \;), \\
& (\; i+3, & i+5, & i+6, & \infty; & 2,7,0,5,7,2 \;); \\
\\
P_{2i}: & (\; i, & i+1, & i+2, & i+5; & 3,5,2,2,8,6 \;), \\
& (\; i+3, & i+4, & i+6, & \infty; & 6,5,1,8,4,5 \;); \\
\\
P_{3i}: & (\; i, & i+1, & i+2, & i+4; & 4,3,7,8,3,4 \;), \\
& (\; i+3, & i+5, & i+6, & \infty; & 6,4,3,7,6,8 \;).
\end{array}$$

This gives an LRB(4, 9, 8).
(ii) $v = 12$, parallel classes: P_{1i}, P_{2i}, P_{3i}, $i \in Z_{11}$:

$$\begin{array}{llllll}
P_{1i}: & (\; i, & i+2, & i+4, & i+5; & 2,5,5,3,3,0 \;), \\
& (\; i+9, & i+10, & i+1, & i+6; & 1,8,2,7,1,3 \;), \\
& (\; i+3, & i+7, & i+8, & \infty; & 2,4,0,2,7,5 \;); \\
\\
P_{2i}: & (\; i+4, & i+8, & i+10, & i; & 3,0,3,6,0,3 \;), \\
& (\; i+7, & i+9, & i+1, & i+2; & 0,6,1,6,1,4 \;), \\
& (\; i+3, & i+5, & i+6, & \infty; & 8,4,1,5,2,6 \;); \\
\\
P_{3i}: & (\; i, & i+1, & i+4, & i+6; & 6,7,8,1,2,1 \;), \\
& (\; i+3, & i+5, & i+7, & i+8; & 5,0,7,4,2,7 \;), \\
& (\; i+9, & i+10, & i+2, & \infty; & 8,4,3,5,4,8 \;).
\end{array}$$

This gives an LRB(4, 9, 12). $\square$

Lemma 2.2. *There exists an* LRB$(4,9;20)$.

Proof. Let $X = Z_{19} \cup \{\infty\}$ with ordering $0 < 1 < 2 < \cdots < 18 < \infty$.
Parallel classes: P_{1i}, P_{2i}, P_{3i}, $i \in Z_{19}$:

$$\begin{array}{llllll}
P_{1i}: & (\ i+1, & i+3, & i+9, & i+16; & 0,6,6,6,6,0\ ),\\
 & (\ i+2, & i+5, & i+6, & i+8; & 0,0,1,0,1,1\ ),\\
 & (\ i+13, & i+14, & i+18, & i+4; & 1,7,8,6,7,1\ ),\\
 & (\ i+7, & i+10, & i+15, & i+17; & 2,4,6,2,4,2\ ),\\
 & (\ i+11, & i+12, & i, & \infty; & 2,7,0,5,7,2\ );\\
 & & & & & \\
P_{2i}: & (\ i+13, & i+18, & i+2, & i+6; & 0,3,7,3,7,4\ ),\\
 & (\ i+4, & i+10, & i+12, & i+16; & 2,5,1,3,8,5\ ),\\
 & (\ i+7, & i+8, & i+9, & i+17; & 3,7,0,4,6,2\ ),\\
 & (\ i+11, & i+14, & i+15, & i+1; & 5,1,5,5,0,4\ ),\\
 & (\ i, & i+3, & i+5, & \infty; & 7,3,6,5,8,3\ );\\
 & & & & & \\
P_{3i}: & (\ i+3, & i+8, & i+9, & i+10; & 8,5,3,6,4,7\ ),\\
 & (\ i+15, & i+18, & i+5, & i+6; & 4,2,1,7,6,8\ ),\\
 & (\ i+12, & i+16, & i+1, & i+4; & 8,1,0,2,1,8\ ),\\
 & (\ i+2, & i+7, & i+11, & i+13; & 6,4,1,7,4,6\ ),\\
 & (\ i+14, & i+17, & i, & \infty; & 6,5,1,8,4,5\ ).
\end{array}$$

This gives an LRB$(4,9,20)$. □

Lemma 2.3. *There exists an* LRB$(4,9;24)$.

Proof. Let $X = Z_{23} \cup \{\infty\}$ with ordering $0 < 1 < 2 < \cdots < 22 < \infty$.
Parallel classes: P_{1i}, P_{2i}, P_{3i}, $i \in Z_{23}$:

$$\begin{array}{llllll}
P_{1i}: & (\ i, & i+2, & i+9, & i+13; & 1,2,4,1,3,2\ ),\\
 & (\ i+17, & i+18, & i+20, & i+5; & 2,7,5,5,3,7\ ),\\
 & (\ i+4, & i+10, & i+15, & i+19; & 0,4,3,4,3,8\ ),\\
 & (\ i+14, & i+16, & i+21, & i+22; & 0,5,4,5,4,8\ ),\\
 & (\ i+3, & i+6, & i+7, & i+12; & 3,1,0,7,6,8\ ),\\
 & (\ i+1, & i+8, & i+11, & \infty; & 8,7,2,8,3,4\ );\\
 & & & & & \\
P_{2i}: & (\ i+18, & i, & i+3, & i+4; & 0,1,5,1,5,4\ ),\\
 & (\ i+10, & i+11, & i+13, & i+17; & 6,4,7,7,1,3\ ),\\
 & (\ i+7, & i+8, & i+15, & i+20; & 1,5,8,4,7,3\ ),\\
 & (\ i+19, & i+21, & i+5, & i+9; & 3,6,1,3,7,4\ ),\\
 & (\ i+1, & i+6, & i+12, & i+4; & 2,1,5,8,3,4\ ),\\
 & (\ i+16, & i+22, & i+2, & \infty; & 5,1,1,5,5,0\ );
\end{array}$$

$$\begin{array}{lllllll}
P_{3i}\colon & (& i, & i+6, & i+8, & i+13; & 3,2,0,8,6,7\ ),\\
 & (& i+11, & i+20, & i+22, & i+3; & 7,0,0,2,2,0\ ),\\
 & (& i+7, & i+14, & i+16, & i+17; & 2,8,2,6,0,3\ ),\\
 & (& i+10, & i+15, & i+18, & i+19; & 6,8,4,2,7,5\ ),\\
 & (& i+1, & i+2, & i+5, & i+12; & 0,6,6,6,6,0\ ),\\
 & (& i+21, & i+4, & i+9, & \infty; & 7,8,6,1,8,7\ ).
\end{array}$$

This gives an LRB$(4,9,24)$. □

Lemma 2.4. *There exists an* RGD$(4,9;36t)$ *if* $t\in\{2,3,5,6\}$.

Proof. The conclusion follows from Lemmas 2.1–2.3 and Theorem 2.1. □

3. Recursive constructions

Let $(X,\mathbf{G},\mathbf{A})$ be an RGD$(K,M;v)$ and $(Y,\mathbf{H},\mathbf{B})$ be an RGD$(K,M;u)$. If $X\subset Y$, $\mathbf{G}\subset\mathbf{H}$, and each parallel class of $\mathbf{A}$ is a part of some parallel class of $\mathbf{B}$, then $(X,\mathbf{G},\mathbf{A})$ is called a sub-RGDD of $(Y,\mathbf{H},\mathbf{B})$ or $(X,\mathbf{G},\mathbf{A})$ is embedded in $(Y,\mathbf{H},\mathbf{B})$.

In this section, we will give recursive constructions for $\{4\}$-RGDDs containing sub-RGDDs.

Theorem 3.1 (Shen [8]). *If there exist an* RGD$(4,m;4v)$ *and an* RTD$(4,v)$, *then there exists an* RGD$(4,m;4(3s+1)v)$ *containing a sub*-RGD$(4,m;4v)$ *for any integer* $s\geqslant 0$.

Theorem 3.2 (Shen [8]). *If there exists an* RGD$(4,m;v)$ *and* $t\notin\{2,3,6,10\}$, *then there exists an* RGD$(4,tm;tv)$.

Let $(X,\mathbf{G},\mathbf{A})$ *be a* K-GDD *of type* T. *A holey parallel class with hole* G *is a subset* P *of* $\mathbf{A}$ *which partitions* $X\backslash G$ *for some* $G\in\mathbf{G}$. $(X,\mathbf{G},\mathbf{A})$ *is called a Kirkman* K-*frame of type* T *if* $\mathbf{A}$ *can be partitioned into holey parallel classes.*

For the application of Kirkman frames in the construction of RGDDs with sub-RGDDs, we have the following frame constructions.

Lemma 3.1. *If there is a Kirkman* $\{4\}$-*frame of type* $\prod_{1\leqslant i\leqslant s}t_i^{u_i}$ *such that there exists an* RGD$(4,m;t_i+\varepsilon)$ *containing a sub*-RGD$(4,m;\varepsilon)$ for $1\leqslant i\leqslant s$, *then there exists an* RGD$(4,m;v+\varepsilon)$ *containing a sub*-RGD$(4,m;t_i+\varepsilon)$ *for each* i, $1\leqslant i\leqslant s$, *where* $v=\sum_{1\leqslant i\leqslant s}t_iu_i$.

Proof. There are precisely $(v-m)/3$ parallel classes in an RGD$(4,m;v)$ and there are $|G|/3$ holey parallel classes with hole G in a Kirkman $\{4\}$-frame for each group G of the frame. Let $(X,\mathbf{G},\mathbf{A})$ be a Kirkman $\{4\}$-frame of type $\prod_{1\leqslant i\leqslant s}t_i^{u_i}$. Let $(X_0,\mathbf{G}_0,\mathbf{A}_0)$ be an RGD$(4,m;\varepsilon)$ and let the $r=(\varepsilon-m)/3$ parallel classes be denoted $P_{0,1},P_{0,2},\ldots,P_{0,r}$.

For each $G \in \mathbf{G}$ with $|G| = t_i$, form an RGD$(4, m; t_i + \varepsilon)$ on the set $G \cup X_0$ containing $(X_0, \mathbf{G}_0, \mathbf{A}_0)$ as a subdesign. Let the group-set be $\mathbf{H}(G) \cup \mathbf{G}_0$. There are $(t_i + \varepsilon - m)/3$ parallel classes in the RGD$(4, m; t_i + \varepsilon)$, among which $(\varepsilon - m)/3$ containing a parallel class of $\mathbf{A}_0$. Let $P_{0,j}(G)$ denote the parallel class containing $P_{0,j}(G)$, $1 \leqslant j \leqslant r$, and let the remaining parallel classes be $Q_j(G)$, $1 \leqslant j \leqslant t_i/3$. Let the $t_i/3$ holey parallel classes of the frame with hole G be $P_i(G)$, $1 \leqslant i \leqslant t_i/3$. Now, let

$$Y = X \cup X_0,$$

$$\mathbf{H} = \mathbf{G}_0 \cup \left\{ \bigcup_{G \in \mathbf{G}} \mathbf{H}(G) \right\},$$

$$\mathbf{B} = \left\{ \bigcup_{G \in \mathbf{G}} \bigcup_{1 \leqslant j \leqslant |G|/3} \{P_j(G) \cup Q_j(G)\} \right\}$$

$$\bigcup \left\{ \bigcup_{1 \leqslant j \leqslant r} \left\{ \bigcup_{G \in \mathbf{G}} P_{0,j}(G) \setminus P_{0,j} \right\} \bigcup P_{0,j} \right\}.$$

Then $(Y, \mathbf{H}, \mathbf{B})$ is an RGD$(4, m; v + \varepsilon)$ containing $(X_0, \mathbf{G}_0, \mathbf{H}_0)$ as a subdesign and containing an RGD$(4, m; |G| + \varepsilon)$ as a subdesign. This completes the proof. $\square$

The following lemma is a generalization of Lemma 3.1.

Lemma 3.2. *If there is a Kirkman* $\{4\}$*-frame of type* $t_0 \prod_{1 \leqslant i \leqslant s} t_i^{u_i}$ *such that there exists an* RGD$(4, m; t_i + \varepsilon)$ *containing a sub*-RGD$(4, m; \varepsilon)$ *for* $1 \leqslant i \leqslant s$, *and there exists an* RGD$(4, m; t_0 + \varepsilon)$, *then there exists an* RGD$(4, m; v + \varepsilon)$ *containing a sub*-RGD $(4, m; t_0 + \varepsilon)$, *where* $v = \sum_{1 \leqslant i \leqslant s} t_i u_i + t_0$.

In order to apply the frame constructions for RGDDs, we need the following two basic constructions for Kirkman $\{4\}$-frames which can be found in [6].

Lemma 3.3. *If there exist a Kirkman* $\{4\}$*-frame of type* t^u *and an* RTD$(4, m)$, *then there exists a Kirkman* $\{4\}$*-frame of type* $(mt)^u$.

Lemma 3.4. *Let* $(X, \mathbf{G}, \mathbf{A})$ *be a* GDD. *Let* $\omega : X \to Z^+ \cup \{0\}$ *be a weight function on* X. *Suppose that for each* $B \in \mathbf{A}$ *there exists a Kirkman* $\{4\}$*-frame of type* $\{\omega(x) \colon x \in B\}$. *Then there exists a Kirkman* $\{4\}$*-frame of type* $\{\sum_{x \in G} \omega(x) \colon G \in \mathbf{G}\}$.

We will also need the following existence results for uniform Kirkman $\{4\}$-frames.

Lemma 3.5 (Colbourn et al. [2]). (i) *For any* $t \geqslant 1$, *there is a Kirkman* $\{4\}$*-frame of type* $(36t - 9)^u$ *if and only if* $u \equiv 1 \pmod 4$ *and* $u \geqslant 5$.

(ii) *If* $t \equiv 0 \pmod{24}$ *then this is a Kirkman* $\{4\}$*-frame of type* t^u *if and only if* $u \geqslant 5$ *with the possible exception* $u = 12$.

(iii) *There is a Kirkman* $\{4\}$*-frame of type* 36^u *if* $u \in \{5,6,7\}$.

The following lemma plays an important role in the construction of RGD$(4,9;v)$s:

Lemma 3.6. *If there exist a* TD$(7,3n)$ *and* $0 \leqslant m_1, m_2 \leqslant n$, *then there is a Kirkman* $\{4\}$*-frame of type* $(36n)^5 \cdot (36m_1)^1 \cdot (36m_2)^1$.

Proof. For a given TD$(7,3n)$, delete $3(n-m_1)$ points from the first group and delete $3(n-m_2)$ points from the second group. This gives a $\{5,6,7\}$-GDD of type $(3n)^5 \cdot (3m_1)^1 \cdot (3m_2)^1$. Give each point weight 12. Since there exists a Kirkman $\{4\}$-frame of type 12^u for each $u \in \{5,6,7\}$ by Lemma 3.5(ii), the conclusion then follows from Lemma 3.4. $\square$

4. Existence of RGD(4, 9; v)

In this section, we will give an almost complete solution to the existence of RGD $(4,9;v)$s. Let

$$S(9) = \{t\text{: There exists an RGD}(4,9;36t)\}.$$

$$S^*(9) = \{t\text{: There is an RGD}(4,9,36t) \text{ containing a sub-RGD}(4,9;36)\}.$$

Lemma 4.1. *If* $t \in S(9)$, *then* $(4t-1)s + t \in S(9)$ *for all* $s \geqslant 0$. *If* $t \in S^*(9)$, *then* $(4t-1)s + t \in S^*(9)$.

Proof. Obviously, there is an RGD$(4,9;36)$ which is in fact an RTD$(4,9)$. Since there is a Kirkman $\{4\}$-frame of type $(36t-9)^{4s+1}$ for any $t,s \geqslant 1$, by Lemma 3.5(i), then the conclusion follows from Lemma 3.1 with $\varepsilon = 9$. $\square$

Lemma 4.2. *If* $t \in S(9)$, *then* $(3s+1)t \in S(9)$ *for all* $s \geqslant 1$. *If* $t \in S^*(9)$, *then* $(3s+1)t \in S^*(9)$.

Proof. By Lemma 1.2, there is an RGD$(4,1;4(3s+1))$ for each $s \geqslant 0$. Let $(X,\mathbf{A})$ be an RGD$(4,1;4(3s+1))$ and let the parallel classes be $P_0, P_1, \ldots, P_{4s}$. For each $x \in X$, let $S(x) = \{x_1, x_2, \ldots, x_{9t}\}$. For each block $B \in F_0$, form an RGD$(4,9;36t)$ on the set $\bigcup_{x\in B} S(x)$. Let $\mathbf{G}(B)$ be the group set and let $P_{0,1}(B), P_{0,2}(B), \ldots, P_{0,4t-3}(B)$ be the parallel classes of the RGD$(4,9;36t)$. Let

$$\mathbf{G} = \bigcup_{B\in F_0} \mathbf{G}(B), \quad P_{0j} = \bigcup_{B\in F_0} P_{0i}(B), \ \ 1 \leqslant j \leqslant 4t-3.$$

For each $B \in P_i$, $1 \leqslant i \leqslant 4s$, form an RTD$(4,9t)$ on the set $\bigcup_{x\in B} S(x)$ with groups $S(x)$ where $x \in B$. Let $P_{i1}(B), P_{i2}(B), \ldots, P_{i,9t}(B)$ be the parallel classes. Let

$$P_{ij} = \bigcup_{B \in P_i} P_{ij}(B), \quad 1 \leqslant j \leqslant 9t.$$

Now, let

$$Y = \bigcup_{x \in \mathbf{X}} S(x),$$

$$\mathbf{B} = \bigcup_{1 \leqslant j \leqslant 4t-3} P_{0j} \cup \left\{ \bigcup_{1 \leqslant i \leqslant 4s} \left\{ \bigcup_{1 \leqslant j \leqslant 9t} P_{ij} \right\} \right\}.$$

Then $(Y, \mathbf{G}, \mathbf{B})$ is an RGD$(4,9;36t \cdot (3s+1))$ and so $(3s+1)t \in S(9)$. Obviously if $t \in S^*(9)$, then $(3s+1)t \in S^*(9)$. This completes the proof. $\square$

Lemma 4.3. $t \in S^*(9)$ *for each* $t \equiv 0 \,(\mathrm{mod}\, 4)$, $t \geqslant 8$, $t \neq 88, 124$.

Proof. By Lemma 1.4, for each $t \equiv 0 \,(\mathrm{mod}\, 4)$, $t \geqslant 8$ and $t \neq 88, 132$, there exists an RGD$(4,3;3t)$. Then there exists an RGD$(4,36;36t)$ by Theorem 3.2. Replace each group of the RGD$(4,36;36t)$ by an RGD$(4,9;36)$ gives an RGD$(4,9;36t)$ containing a sub-RGD$(4,9;36)$. This completes the proof. $\square$

Lemma 4.4 (Rees and Stinson [6]). *If there is an* RGD$(k,m;mu)$, *a Kirkman* $\{k\}$-*frame of type* $(tm)^v$, *where* $u \geqslant t+1$, *and an* RTD(k,tv), *then there exists an* RGD $(k,tm;tmuv)$.

Lemma 4.5. *If* $t \in \{15, 18, 27\}$, *then* $t \in S(9)$.

Proof. In Lemma 4.4, let $t = m = 3$, $u = 12$, $v = 5$ or 9, then we have $15 \in S(9)$ and $27 \in S(9)$. Let $t = m = 3$, $u = 8$ and $v = 9$, then we have $18 \in S(9)$. $\square$

Lemma 4.6. $17 \in S(9)$.

Proof. Let $n = 3$, $m_1 = 0$ and $m_2 = 1$ in Lemma 3.6, then there is a Kirkman $\{4\}$-frame of type $(3 \cdot 36)^5 \cdot 0^1 \cdot 36^1$. Since there exists an RGD$(4,9;72)$ by Lemma 2.4 and there exists an RGD$(4,9,144)$ containing an RGD$(4,9;36)$, then by Lemma 3.2, there exists an RGD$(4,9;17 \cdot 36)$ and so $17 \in S(9)$. $\square$

Lemma 4.7. *If* $1 \leqslant t \leqslant 35$, $t \neq 11$, *then* $t \in S(9)$.

Proof. The conclusion follows from Lemmas 2.4 and 4.1–4.5. $\square$

Lemma 4.8. $t \in S(9)$ *for all* $t \geqslant 36$.

Proof. Let $n=7$ in Lemma 3.6, then we have a Kirkman $\{4\}$-frame of type $(7\cdot 36)^5\cdot(36m_1)^1\cdot(36m_2)^1$ for $0\leqslant m_2\leqslant 7$, then we have $t\in S(9)$ for $36\leqslant t\leqslant 50$. Then let $n=9,12,15,19$ and $3n_1$ for $n_1\geqslant 7$. It can be checked that there is a TD$(7,3n)$. Now, let $m_1\equiv 0\,(\mathrm{mod}\,3)$ and $0\leqslant m_1\leqslant n$, $0\leqslant m_2\leqslant n$ and $m_2\neq 11$ in Lemma 3.6. Since $t\in S^*(9)$ for $t=n+1$ and m_1+1 by Lemmas 4.2, 4.3 and $s\in S(9)$ for $s=m_2+1$, the conclusion then follows. □

Combining Lemmas 4.7 and 4.8, we have proved the following theorem.

Theorem 4.1. *There exists an* RGD$(4,9;v)$ *if and only if* $v\equiv 0\,(\mathrm{mod}\,36)$, *with the possible exception of* $v=396$.

5. Existence of RGD(4, *m*, *v*) with *m* odd

In this section, we will prove that the necessary conditions (1,2) are also sufficient for the existence of an RGD$(4,m;v)$ if $m\equiv 1\,(\mathrm{mod}\,2)$ and $m\neq 3,9$.

Theorem 5.1. *If* $m\equiv 1,5,7,11\,(\mathrm{mod}\,12)$, *then there exists an* RGD$(4,m;v)$ *if and only if*

$$v\equiv 4m\,(\mathrm{mod}\,12m).$$

Proof. The conclusion follows from Lemmas 1.1, 1.2 and Theorem 3.2. □

Theorem 5.2. *If* $m\equiv 3,9\,(\mathrm{mod}\,12)$, $m\neq 3,9$, *then there exists an* RGD$(4,m;v)$ *if and only if*

$$v\equiv 0\,(\mathrm{mod}\,4m).$$

Proof. Let $m=3m_1$, then $m_1\equiv 1\,(\mathrm{mod}\,2)$ and $m_1\geqslant 5$, and so there exists an RTD$(4,m_1)$. It follows from Lemma 1.4 and Theorem 3.2 that there exists an RGD$(4,m;v)$ if $v\equiv 4m\,(\mathrm{mod}\,12m)$ and $v\notin\{88m,124m\}$. But since $m\equiv 1\,(\mathrm{mod}\,2)$ and $m\geqslant 5$, then there exists an RGD$(4,m;4m)$.

Since there are an RGD$(4,1,88)$ and an RGD$(4,1;124)$ by Lemma 1.2, then there exist an RGD$(4,m;88m)$ and an RGD$(4,m;124m)$ by Theorem 3.2.

Combining Lemma 1.4, Theorems 4.1 and 5.1 and 5.2, we have proved our main theorem. □

Theorem 5.3. *If* $m\equiv 1\,(\mathrm{mod}\,2)$, *then there exists an* RGD$(4,m;v)$ *if and only if* $v\geqslant 4m$ *and*

(i) $v\equiv 4m\,(\mathrm{mod}\,12m)$ *if* $m\equiv 1,5,7,11\,(\mathrm{mod}\,12)$,

(ii) $v\equiv 0\,(\mathrm{mod}\,4m)$ *if* $m\equiv 3,9\,(\mathrm{mod}\,12)$

with the exception $(v,m)=(12,3)$ *and* 3 *possible exceptions, where* $(v,m)=(264,3)$, $(372,3)$ *and* $(396,9)$.

6. Existence of RGD(4, *m*; *v*) with $m \equiv 4$ or $8 \pmod{12}$

In this section, we will prove that the necessary conditions (1) are also sufficient for the existence of an $\mathrm{RGD}(4,m;v)$ if $m\equiv 4$ or $8 \pmod{12}$ with two possible exceptions.

Lemma 6.1. *If* $m\equiv 4$ *or* $8 \pmod{12}$ *and* $m\neq 8$, *then there exists an* $\mathrm{RGD}(4,m;v)$ *if and only if*

$$v\equiv m \pmod{3m}$$

Proof. The conclusion follows from Lemma 1.1, 1.2 and Theorem 3.2. □

Lemma 6.2. *If* $v\equiv 8 \pmod{24}$, $v\neq 80$ *or* 104, *then there exists an* $\mathrm{RGD}(4,8;v)$.

Proof. Obviously, there is an $\mathrm{RGD}(4,8;32)$ containing an $\mathrm{RGD}(4,8;8)$. By Lemma 3.5(ii), there exists an $\mathrm{RGD}(4,8;24u+8)$ for each $u\geqslant 5$, $u\neq 12$. An $\mathrm{RGD}(4,8;56)$ can be found in [6]. Since there is a Kirkman $\{4\}$-frame of type 48^6 by Lemma 3.5(iii), then there exists an $\mathrm{RGD}(4,8;48\cdot 6+8)$. This completes the proof. □

Combining Lemmas 6.1 and 6.2 gives the following theorem.

Theorem 6.1. *If* $m\equiv 4$ *or* $8 \pmod{12}$, *then there exists an* $\mathrm{RGD}(4,m;v)$ *if and only if*

$$v\equiv m \pmod{3m}$$

with two possible exceptions where $(v,m)=(80,8)$, $(104,8)$ $(400,40)$ *or* $(520,40)$.

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