In the last class we saw some examples of Linear Programming (LP) problems. In this class, we will first present the canonical and standard forms of LP problems and then discuss a few algorithms to solve the problems.

24.1 Standard and Canonical Forms of Linear Programming (LP) Problems:

Here is a typical example of an LP problem:

\[
\begin{align*}
\text{Minimize} & \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
\text{such that} & \quad \begin{cases} 
2x_1 + x_2 + x_3 \leq 4 \\
x_1 \leq 2 \\
x_3 \leq 3 \\
3x_2 + x_3 \leq 6 \\
x_1, x_2, x_3 \geq 0
\end{cases}
\end{align*}
\]  

\quad (24.1)

Any LP problem can be reduced to the \textit{standard form}, the general representation of which is shown below:

\[
\begin{align*}
\text{Minimize} & \quad \mathbf{C} \cdot \mathbf{x} \\
\text{such that} & \quad \mathbf{A} \cdot \mathbf{x} = \mathbf{b} \\
& \quad \mathbf{x} \geq \mathbf{0}
\end{align*}
\]  

\quad (24.2)

where \( \mathbf{A} \) is an \( m \times n \) matrix, \( \mathbf{x} \) and \( \mathbf{c} \) are vectors of length \( n \), and \( \mathbf{b} \) is a vector of length \( m \). The LP problem represented by equation 24.1 can be reduced to the standard form by adding a \textit{surplus} variable to each inequality as shown in the following representation:

\[
\begin{align*}
\text{Minimize} & \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
\text{such that} & \quad \begin{cases} 
2x_1 + x_2 + x_3 + x_4 = 4 \\
x_1 + x_5 = 2 \\
x_3 + x_6 = 3 \\
3x_2 + x_3 + x_7 = 6 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{cases}
\end{align*}
\]  

\quad (24.3)

Equation 24.1 is expressed in what is called the \textit{canonical form}. A general canonical form representation of an LP problem is presented in equation 24.4.

\[
\begin{align*}
\text{Minimize} & \quad \mathbf{C}' \cdot \mathbf{x}' \\
\text{such that} & \quad \mathbf{A}' \cdot \mathbf{x}' \leq \mathbf{b}' \\
& \quad \mathbf{x}' \geq \mathbf{0}
\end{align*}
\]  

\quad (24.4)

where \( \mathbf{A}' \) is an \( m \times n' \) matrix, \( \mathbf{x}' \) and \( \mathbf{C}' \) are vectors of length \( n' \), and \( \mathbf{b}' \) is a vector of length \( m \). It is important to note that \( n' \) represents the number of variables in the canonical form and is not the same as the variable \( n \) used in the corresponding standard representation.
24.2 Assumptions

The following assumptions are made on the standard form of LP problems:

1. Rows of $A$ are linearly independent: This guarantees that there are no redundancies or inconsistencies.

2. $m < n$: If $m > n$ we will have redundancies or inconsistencies in the constraint equations; if $m = n$ we will have a unique solution that is easy to solve. We have no further interest in this case.

24.3 Feasible Solutions

In this section topics such as half-spaces, sets of feasible solutions, feasible polyhedra, etc., will be discussed. We will also see how an LP problem expressed in standard form can be converted to its canonical counterpart.

24.3.1 Half spaces and feasible solution sets

**Definition 24.1** The set of feasible solutions of an LP problem $F = \{x \in \mathbb{R}^n : A \cdot x = b$ and $x \geq 0\}$ In other words, the set of feasible solutions represents all possible values of $x$ that satisfy $A \cdot x = b$.

**Definition 24.2** A half-space in $\mathbb{R}^k$ is a set of the form $\{x \in \mathbb{R}^k : \sum_{j=1}^{k} a_j x_j \leq r\}$ where $r$ is a constant.

![Diagram of half-spaces](image)

(a) 1-d Half Space

(b) 2-d Half Space

(c) 3-d Half Space

**Figure 24.1:** Geometrical representation of half-spaces in various dimensions.

Figure 24.1 shows half-spaces geometrically in 1, 2, and 3 dimensions. If we interpret time as the fourth dimension an example of a half-space in 4D would be an instance of the entire universe at a fixed point in time and from that time onward.

**Definition 24.3** A polyhedron is an intersection of a finite number of half spaces in $\mathbb{R}^k$. A polytope is a bounded polyhedron.
Figure 24.2 shows examples, in two dimensions, of a polyhedron and a polytope.

(a) Polyhedron: (Unbounded)  
(b) Polytope

Figure 24.2: Examples of a polyhedron and a polytope.

Claim 24.4 For an LP problem in standard form with \( n \) variables and \( m \) equations, the feasible set can be described as a polyhedron in \( \mathbb{R}^{n-m} \).

Proof: The following is only a general idea of the proof.

In each equation, in the standard form of an LP problem, the value of exactly one variable is fixed by the values of other variables in the equation. Hence fixing the values of \((n-m)\) variables in the entire set of \( m \) equations determines the values of the other \( m \) variables. This subset of \((n-m)\) variables will range in a polyhedron in \( \mathbb{R}^{n-m} \). It is easy to see that any point in this polyhedron will determine the values of the other \( m \) variables. Hence, we can describe the feasible set as a polyhedron in \( \mathbb{R}^{n-m} \).

24.3.2 Converting standard form to canonical form

Claim 24.5 LP problems in standard form with \( m \) equations and \( n \) variables can be converted to canonical form with \( m \) equations and \((n-m)\) variables.

Proof: We know that the rows of \( A \) are linearly independent. From linear algebra it can be inferred that \( A \) also contains \( m \) linearly independent columns. Let us assume without loss of generality that the first \( m \) columns of \( A \) are linearly independent. Let \( B \) be the matrix obtained from the first \( m \)-columns of \( A \), as shown in the following Figure 24.3.

\[
\begin{bmatrix}
A \\
B
\end{bmatrix}
= 
\begin{bmatrix}
X \\
0
\end{bmatrix}
= 
\begin{bmatrix}
b \\
0
\end{bmatrix}
\]

Figure 24.3: Matrix format.
Since all the columns of $\mathbf{B}$ are linearly independent, $\mathbf{B}^{-1}$ must exist. Multiplying the system of equations by $\mathbf{B}^{-1}$, we get the following structure:

$$
\mathbf{A}' = \mathbf{B}' \cdot \mathbf{A}
$$

where $\mathbf{A}' = \mathbf{B}^{-1} \cdot \mathbf{A}$ and $\mathbf{b}' = \mathbf{B}^{-1} \cdot \mathbf{b}$. The system of equations in Figure 24.4 can be equivalently expressed as follows:

$$
x_i + \sum_{j=m+1}^{n} a'_{ij} x_j = b'_i \quad \forall i = 1, \ldots, m
$$

$$
x_i \geq 0 \quad \forall i = 1, \ldots, m
$$

$$
\Rightarrow \sum_{j=m+1}^{n} a'_{ij} x_j \leq b'_i \quad \forall i = 1, \ldots, m
$$

$$
x_j \geq 0 \quad \forall j = m + 1, \ldots, n
$$

The system of inequalities represented by equation 24.6 is clearly in canonical form. This completes the proof. 

24.3.3 Geometrical interpretation of feasible solution sets

Figure 24.5 corresponds to the LP problem we discussed in Section 24.1. In canonical form the set of constraints to this problem is as given in equation 24.1. Equation 24.3 presents the same problem in standard form. In this subsection we will use both forms of the problem to interpret the polyhedron in Figure 24.5.

**Canonical form:** In canonical form each inequality represents a half-space and the intersection of all such half-spaces is the polyhedron in Figure 24.5. Each face in the polyhedron represents one of the inequalities. More precisely, equality corresponds to the face and inequality to the side of the face that contains the polyhedron.

**Standard form:** Each face in the polyhedron in Figure 24.5 represents the case where exactly one of the surplus (in this case) variables vanishes. The value of the surplus variable in an equation at any point inside the polyhedron (representing a feasible solution) is equal to the distance of that point from the face that represents the same equation with that particular surplus variable set to zero. For example, consider the face representing the equation $x_1 + x_2 + x_3 = 0$. All points on the face correspond to the case $x_4 = 0$. For any point inside the polyhedron, $x_4 = 4 - (x_1 + x_2 + x_3)$. 

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
X \\
\end{bmatrix}_n =
\begin{bmatrix}
b' \\
\end{bmatrix}_m
\]

Figure 24.4: Matrix format after multiplication by $\mathbf{B}^{-1}$. 

\[
x_i + \sum_{j=m+1}^{n} a'_{ij} x_j = b'_i \quad \forall i = 1, \ldots, m
\]

\[
x_i \geq 0 \quad \forall i = 1, \ldots, m
\]

\[
\Rightarrow \sum_{j=m+1}^{n} a'_{ij} x_j \leq b'_i \quad \forall i = 1, \ldots, m
\]

\[
x_j \geq 0 \quad \forall j = m + 1, \ldots, n
\]
24.3.4 A vertex in a feasible polyhedron

A vertex in a feasible polyhedron can be described both geometrically and algebraically as follows:

**Geometrical description:** A vertex is an intersection of at least \((n - m)\) hyper-planes. A vertex is called *degenerate* if it is an intersection of more than \((n - m)\) hyper-planes. For example, in three-dimensional space \((n - m = 3)\), a pyramid shaped polyhedron has a degenerate vertex in its apex, which is an intersection of 4 faces of the pyramid. In our problems we will assume that all vertices are *non-degenerate* for simplicity.

**Algebraic description:** In the standard form of an \(n\)-variable LP problem, a vertex corresponds to a feasible solution in which at least \((n - m)\) variables are set to zero.

24.4 Algorithms to Solve LP Problems

**Claim 24.6** If \(F\) is a polytope, then \(C \cdot x\) is minimized at a vertex of \(F\). Otherwise, \(C \cdot x\) can be arbitrarily small (unbounded).

**Proof:** Again, the following explanation is just a proof-idea and is not rigorous.

Let’s consider an LP problem in two-dimensions whose objective function is \(C \cdot x\). Any equality of the form \(C \cdot x = d\), where \(d\) is a constant, is a line in 2D space. As we decrease the value of \(d\) the line moves parallel to itself in a preferred direction as shown in Figure 24.6(a). In three dimensions, the objective function will be a plane. We want to move the line (or plane as the case may be) as far as possible in the preferred direction until the objective function is minimized. The last point in the polyhedron that is contained in such a line (plane) as it is moved, will either be a vertex or an edge that is parallel to the line of constant optimality. If the last point is in an edge, the line still contains at least one vertex as shown in the Figure 24.6(b). Hence the minimum point is a vertex.

If \(F\) is not a polytope then, in some cases, it is possible to achieve arbitrarily small values of the objective function by moving the line of constant optimality indefinitely. For the objective function to be unbounded
it is necessary, but not sufficient, for $F$ to be unbounded. Problem 4(b) in Homework 5 provides an example of a problem where the polyhedron is unbounded but the optimal solution is not.

Figure 24.6: Minimum value of the objective function occurs at a vertex.

### 24.4.1 A simple algorithm

Here is a simple algorithm based on claim 24.6 to finding the optimal solutions to LP problems:

1. For each vertex $x$ of $F$:
   2. Compute $C \cdot x$
   3. Return minimum value found.

This algorithm has the following major drawbacks:

*Finding the vertices:* Computing the vertices is non-trivial.

*Exponential running time:* The number of vertices to be considered can be exponential in input size $n - m$.

To illustrate this, let's consider the example of a $k$-dimensional cube. For $k = 1$, we have a single face and two vertices as shown in Figure 24.7(a). For $k = 2$, we have a square with 4 faces and 4 vertices as in Figure 24.7(b). In a three-dimensional cube, we have 6 faces and 8 vertices. Thus, we notice that on the addition of each dimension, the number of vertices increases by a factor of 2. This is clearly exponential in $k$.

Figure 24.7: Number of vertices in a $k$-dimensional cube.
24.4.2 Simplex algorithm

Here is the idea of Simplex algorithm [D49]:

1. Start at some vertex \( x \)
2. Repeat:
   3. Find a neighboring vertex \( y \) s.t. \( C \cdot y < C \cdot x \)
   4. \( x = y \)
3. until no such \( y \) is found
4. Return \( x \) (or unbounded)

Claim 24.7 If a vertex \( x \) is the optimal solution in its neighborhood (locally optimal), it is also the optimal solution in the entire set of feasible solutions (globally optimal).

Proof: Again, note that what follows is only a proof-idea.

We first state without proof that a polyhedral set of feasible solutions is a convex set. Let vertex \( A \) be a locally optimal solution as found by the Simplex algorithm. Let vertices \( B \) and \( C \) be its neighbors as shown in Figure 24.9. The preferred direction for the plane of constant optimality will be away from \( B \) and \( C \) as shown in the figure. Otherwise, one of \( B \) and \( C \) would have been the local optimal solutions. Now, for any other vertex \( D \) or \( E \) to be the globally optimal solution, it must lie in the half-space defined by the plane of constant optimality on the side away from \( A \). For this to happen, the polyhedral set will have to be non-convex as shown in the figure. This leads us to a contradiction. Hence local optimal is also global optimal.

![Figure 24.9: Local optimal is global optimal](image)

24.4.2.1 Basic feasible solutions

Any vertex in the feasible polyhedron is called a Basic Feasible Solution (BFS). From the algebraic definition of a vertex, we know that a (non-degenerate) vertex corresponds to \( (n - m) \) vertices set to zero. Thus, we can define a BFS by setting \( (n - m) \) variables to zero. In a different perspective, we can also define a BFS by specifying a subset of \( m \) variables that are NOT set to zero. We refer to these vertices as

\[
\hat{x} = [x_{B(1)}, x_{B(2)}, \ldots, x_{B(m)}]
\] (24.6)
The matrix formed by corresponding columns of $\mathbf{A}$, denoted by $\mathbf{A}_{B(1)}, \mathbf{A}_{B(2)}, \ldots, \mathbf{A}_{B(m)}$ is called the basis $\mathbf{B}$ for the BFS. In other words,

$$
\mathbf{B} = [\mathbf{A}_{B(1)} \quad \mathbf{A}_{B(2)} \quad \ldots \quad \mathbf{A}_{B(m)}]
$$

(24.7)

The columns of $\mathbf{B}$ must be linearly independent; otherwise, we won’t have an intersection of $(n - m)$ hyperplanes. Hence $\mathbf{B}$ must have an inverse.

We now have:

$$
\mathbf{B} \cdot \mathbf{x} = \mathbf{b}
$$

$$
\Rightarrow \mathbf{x} = \mathbf{B}^{-1} \cdot \mathbf{b}
$$

References