Extending Alternating-Offers Bargaining in One-to-Many and Many-to-Many Settings

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Abstract

Automating negotiations in markets where multiple buyers and sellers operate is a scientific challenge of extraordinary importance. One-to-one negotiations are classically studied as bilateral bargaining problems, while one-to-many and many-to-many negotiations are studied as auctioning problems. This paper aims at bridging together these two approaches, analyzing agents' strategic behavior in one-to-many and many-to-many negotiations when agents follow the alternating-offers bargaining protocol [22]. First, we extend this protocol, proposing a novel mechanism that captures the peculiarities of these settings. Then, we discuss agents' equilibrium strategies with complete information and we preliminarily explore how uncertainty over reserve prices and deadlines can affect equilibrium strategies. Surprisingly, the computation of the equilibrium for realistic ranges of the parameters in one-to-many settings is reduced to the computation of the equilibrium either in one-to-one settings with uncertainty or in one-to-many settings without uncertainty.

1 Introduction

Automated negotiation is an important research area bridging together economics, game theory, and artificial intelligence. It has received a prominent attention in recent years [14] and its importance is widely acknowledged since intelligent agents that negotiate with each other on behalf of human users are expected to lead to more efficient negotiations [25]. A very common class of negotiation is bargaining, which refers to a situation in which individual agents have the possibility of concluding a mutually beneficial agreement which could not be imposed without all individuals' approval. A bargaining theory is an exploration of the relation between the outcome of bargaining and the characteristics of the situation. Cooperative bargaining theory (axiomatic approach) initiated by Nash [18] is concerned with the outcome of bargaining given the list of properties the outcomes are required to satisfy. In the non-cooperative bargaining theory (strategic approach), the outcome is an equilibrium of an explicit model of the bargaining process. The strategic bargaining has received more attention following Rubinstein's path-breaking work [22].

The focus of this work is on analyzing agents' strategic behavior in one-to-many and many-to-many negotiations in which agents are negotiating with multiple trading partners and, at the same time, are facing competition from trading competitors. The subgame perfect equilibrium is presented and equilibrium properties, such us uniqueness, are discussed. We also provide a preliminary extension to the incomplete information setting where there is agents have incomplete information about the reserve price of an agent. Furthermore, we analyze the reduction of computation in one-to-many settings and many-to-many settings. The main goal of this paper is to begin to understand which factors are affecting agents' bargaining position relative to others and agents' equilibrium bargaining strategies.

The situation where an agent has multiple contracting opportunities and faces competition from other agents widely exists in service-oriented computing [28] and Grid resource management [27]. As an example, consider negotiation management [1] for Collaborating, Autonomous Stream Processing systems (CLASP) [5], which has been designed and prototyped in the context of System S project [13] within IBM Research to enable sophisticated stream processing. There are multiple sites running the System S software, each with their own administration and limited processing capabilities. Considering that a site receives a job. After planning [21], the site finds that using only its local resources, it cannot satisfy all resource requirements of the plan. Then, the site negotiates with other sites to acquire resources needed using its negotiation management component [1]. For each resource, there can be multiple providers and the site concurrently negotiates with different resource providers to construct agreements for these resources. There could be other sites requiring the same resource. Thus, each negotiating site needs to take the resource competition into account. While there has been much experimental work (e.g., [2, 19, 26]) on one-to-many and many-to-many negotiations in which an agent synchronously negotiates with multiple agents in discrete time, to our best knowledge, this paper is the first work to provide a game theoretical analysis of agents' strategic interactions in concurrent one-to-many and many-to-many negotiations. The analysis can provide some suggestions for designing negotiation agents in practical electronic marketplaces in which agents are involved in many-to-many negotiations.

In the bargaining theory literature, most work focuses on bilateral bargaining. A variety of negotiation aspects like information and outside options have been studied. Hoverer, one-to-many and many-to-many negotiations are also very important and widely exist in many application domains like e-commerce as well as in human society. For example, when a human being wants to buy a used car, he can negotiate with several dealers concurrently. But he also faces competition due to the existence of other buyers. For one-to-many negotiation, an auction is widely used and, for many-to-many negotiation, market mechanisms like matching or two-sided auction seem more intuitively appropriate. Even if an agent interacts with many agents, a common assumption in this literature is that an agent can pursue only one negotiation at a time. The result is that an agent may terminate a current negotiation in disagreement, in spite of possible gains from trade in order to pursue a more attractive outside alternative. Therefore, the presumption that an agent can pursue only one negotiation at a time appears to be restrictive.

The difference between negotiation and market mechanisms, especially auction, is blurred with the arrival of the Internet and electronic commerce [15]. Negotiation has been treated as a key component of e-commerce and has been applied to e-commerce, manufacturing planning, and distributed vehicle routing. While auction is the most widely implemented and discussed mechanism, only recently the complex, multidimensional, and combinatorial auctions have gained the interest of researchers and foremost practitioners. Negotiations have been somewhat neglected as a possible market mechanism. The proliferation and acceptance of web and Internet technologies made the replacement of some negotiated transactions with auctions not only possible but also efficient. Negotiation-based mechanisms however, still remain the preferred choice when the good and service attributes are ill defined and there are criteria other than price (e.g., reputation, trust, relation and future contracts) [12]. In addition, no third party like auctioneer is needed in bargaining. Strategic agents may prefer bargaining as they can exploit other agents by using learning, collusion, and other bargaining techniques. In this paper, we compared our model with some other mechanisms like VCG auction.

In this paper, negotiating agents make offers following the alternating-offers protocol, which was first analyzed by Rubinstein [22] in a setting with complete information. Of particular interest is the alternating-offers game as the time between offers goes to zero, because this strategic game represents a very general bargaining rule: at any time, a bargainer may make a new offer or accept the most recent offer of his opponent. The alternating-offers protocol captures the most important features of bargaining: bargaining consists of a sequence of offers and decisions to accept or reject these offers. The alternating-offers protocol has been widely used in the bargaining theory literature, e.g., [11, 23, 24], just to name a few.

A central research topic in bargaining theory is understanding bargaining power, which is related to the relative abilities of agents in a situation to exert influence over each other. In bilateral bargaining, each agent's bargaining power is affected by its reserve price, patience attitude, deadline, etc. When many buyers and sellers are involved in negotiation, it is important to investigate how the market competition will affect agents' equilibrium bargaining strategies. With a large number of buyers and sellers, a single agent is unlikely to have much influence on the market equilibrium. Our analysis shows that both bargaining order and market competition affect agents' bargaining power. An agent's bargaining power increases with the number of trading partners (agents of a different type) and decreases with the number of trading competitors.

One crucial challenge in bargaining theory is the development of algorithmic techniques to find equilibria in presence of information incompleteness [3, 11]. While this paper assumes information completeness, we also discuss how to consider incomplete information in one-to-many negotiation and many-to-many negotiation. More specifically, we consider two-type uncertainty about the reserve price of an agent while the reserve prices of other agents are common knowledge. The appropriate solution concept for such a class of game with incomplete information is *sequential equilibrium* [16], specifying a pair: a *system of beliefs* that prescribes how agents' beliefs must be updated during the game and *strategies* that prescribe how agents should act. In a sequential equilibrium there is a sort of circularity between belief system and strategies: strategies must be *sequentially rational* given the belief system and belief system must be *consistent* with respect to strategies. The study of bargaining with uncertain information is well known to be a challenging problem because of this circularity. One contribution of this paper is the development of a novel algorithm to find a pure strategy sequential equilibrium in the setting we study. Our algorithm combines together game theoretical analysis with state space search techniques and it is sound and complete.

The assumptions made in this paper are not more restrictive than related work in the literature. The assumption of the existence of deadline and reserve price in bargaining is widely used in the literature (e.g., [9, 11, 20, 23]). Computing agents' equilibrium strategies in incomplete information bargaining is extremely difficult and most related work only considers one type of uncertainty. For instance, Rubinstein [23] considered bilateral bargaining with uncertainty over two possible discount factors. Gatti *et al.* [11] analyzed bilateral bargaining with one-sided uncertain deadlines. In this paper, we consider the uncertain information about the reserve price of an agent while assuming complete information of other negotiation parameters. As in most related work, we consider the negotiation over a single issue, price of a good. However, our analysis can be easily extended to the multi-attribute negotiations in which the attributes are negotiated simultaneously.

While many-to-many negotiation is a generalization of one-to-many negotiation and bilateral negotiation, we start from the simplest model and then iteratively consider more complex negotiation models. In this way, it is easier for us to understand the factors affecting agents' bargaining power. The rest of this paper proceeds as follows: We start with bilateral negotiation in Section 2. Section 3 discusses one-to-many negotiation and Section 4 investigates many-to-many negotiation. Section 5 discusses how to handle uncertainty of agents' reserve prices. Section 6 analyzes the impact of other types of uncertain information. Section 7 concludes this paper and outlines future research directions.

2 Bilateral Alternating-Offers Negotiation

We follow [11] to describe the non-cooperative bargaining problem between a buyer $\mathbf{b}$ and a seller $\mathbf{s}$. All the agents enter the market at time 0. The seller agent wants to sell a single indivisible good for some quantity of a divisible good ("money"). The buyer agent wants to buy the indivisible good provided by the seller. The characteristics of a transaction that are relevant to an agent are the price x and the number of periods t after the agent's entry into the market that the transaction is concluded.

We study a discrete time (indexed by integers $0, 1, 2, \dots$) bilateral negotiation in this section. A finite horizon alternating-offers bargaining protocol is utilized for the negotiation on one continuous issue (price of a good). Formally, the buyer $\mathbf{b}$ and the seller $\mathbf{s}$ can act at times $t \in \mathbb{N}$. The player function $\iota : \mathbb{N} \to \{\mathbf{b}, \mathbf{s}\}$ returns the agent that acts at time t and is such that $\iota(t) \neq \iota(t+1)$, i.e., a pair of agents bargain by making offers in alternate fashion. For ease of analysis, this paper focuses on single-issue negotiation rather than multiple-issue negotiation. However, our model can be easily extended to handle multi-issue negotiation as in [11].

Possible actions $\sigma^t_{\iota(t)}$ of agent $\iota(t)$ at any time point $t > 0$ are: 1) $\mathit{offer}[x]$, where $x \in \mathbb{R}$ is the proposed price for the good; 2) $exit$, which implies that negotiation between $\mathbf{b}$ and $\mathbf{s}$ fails; and 3) $accept$, which implies that $\mathbf{b}$ and $\mathbf{s}$ make an agreement. At time point $t = 0$ the only allowed actions are 1) and 2). If $\sigma^t_{\iota(t)} = accept$ the bargaining stops and the outcome is (x, t), where x is the value such that $\sigma^{t-1}_{\iota(t-1)} = \mathit{offer}[x]$. This is to say that the agents agree on the value x at time point t. If $\sigma^t_{\iota(t)} = exit$ the bargaining stops and the outcome is $NoAgreement$. Otherwise the bargaining continues to the next time point.

Each agent $\mathbf{a} \in \{\mathbf{b}, \mathbf{s}\}$ has a utility function $U_{\mathbf{a}} : (\mathbb{R} \times \mathbb{N}) \cup NoAgreement \to \mathbb{R}$, which represents its gain over the possible bargaining outcomes. Each utility function $U_{\mathbf{a}}$ depends on $\mathbf{a}$'s reserve price $\mathrm{RP}_{\mathbf{a}} \in \mathbb{R}^+$, temporal discount factor $\delta_{\mathbf{a}} \in (0, 1]$, and deadline $T_{\mathbf{a}} \in \mathbb{N}, T_{\mathbf{a}} > 0$. For ease of analysis, we assume that agents have different reserve prices throughout this paper.

If the outcome of the bargaining is (x, t), then the utility function $U_{\mathbf{a}}$ is defined as:

$$U_{\mathbf{a}}(x,t) = \begin{cases} (\mathrm{RP}_{\mathbf{a}} - x) \cdot \delta^t_{\mathbf{a}} & \text{if } t \leq T_{\mathbf{a}} \text{ and } \mathbf{a} \text{ is a buyer} \\ (x - \mathrm{RP}_{\mathbf{a}}) \cdot \delta^t_{\mathbf{a}} & \text{if } t \leq T_{\mathbf{a}} \text{ and } \mathbf{a} \text{ is a seller} \\ -\epsilon & \text{otherwise} \end{cases}$$

If the outcome is $NoAgreement$, then $U_{\mathbf{a}}(NoAgreement) = 0$. Notice that the assignment of a strictly negative value (we have chosen by convention the value $\epsilon > 0$) to $U_{\mathbf{a}}$ after agent $\mathbf{a}$'s deadline allows one to capture the essence of the deadline: an agent, after its deadline, strictly prefers to exit the negotiation rather than to reach any agreement. Finally, we assume the feasibility of the problem, i.e., $\mathrm{RP}_{\mathbf{b}} \geq \mathrm{RP}_{\mathbf{s}}$, and the rationality of the agents, i.e., each agent will act to maximize its utility. $[\mathrm{RP}_{\mathbf{s}}, \mathrm{RP}_{\mathbf{b}}]$ is the zone of potential agreements.

With complete information the appropriate solution concept for the game we are dealing with is the subgame perfect equilibrium. In subgame perfect equilibrium, agents' strategies are in equilibrium in every possible subgame. Such a solution can be found by backward induction [11].

Initially, it is determined the time point T where the game rationally stops: it is $T = \min(T_{\mathbf{b}}, T_{\mathbf{s}})$. The equilibrium outcome of every subgame starting from $t \geq T$ is $NoAgreement$, since at least one agent will make $exit$. Therefore, at $t = T$ agent $\iota(T)$ would accept any offer x which gives it a utility not worse than $NoAgreement$, namely, any offer x such that $U_{\iota(T)}(x, T) \geq 0$. From $t = T - 1$ back to $t = 0$ it is possible to find the optimal offer agent $\iota(t)$ can make at t, if it makes an offer, and the offers that it would accept. $x^*(t)$ denotes the optimal offer of agent $\iota(t)$ at t. $x^*(t)$ is the offer such that, if $t < T - 1$, agent $\iota(t + 1)$ is indifferent at $t + 1$ between accepting it and rejecting it to make its optimal offer $x^*(t + 1)$ and, if $t = T - 1$, agent $\iota(t + 1)$ is indifferent at $t + 1$ between accepting it and making $exit$. Formally, $x^*(t)$ is such that $U_{\iota(t+1)}(x^*(t)), t) = U_{\iota(t+1)}(x^*(t + 1)$, t+1) if $t < T - 1$ and $U_{\iota(t+1)}(x^*(t), t) = 0$ if $t = T - 1$. The offers agent $\iota(t)$ would accept at t are all those offers that give it a utility no worse than the utility given by offering $x^*(t)$. The equilibrium strategy of any sub-game starting from $0 \leq t < T$ prescribes that agent $\iota(t)$ offers $x^*(t)$ at t and agent $\iota(t + 1)$ accepts it at $t + 1$.

Backward propagation is used to provide a recursive formula for $x^*(t)$: given value x and agent $\mathbf{a}$, we call backward propagation of value x for agent $\mathbf{a}$ the value y such that $U_{\mathbf{a}}(y, t-1) = U_{\mathbf{a}}(x, t)$; we employ the arrow notation $x_{\leftarrow\mathbf{a}}$ for backward propagations. Formally, $x_{\leftarrow\mathbf{b}} = \mathrm{RP}_{\mathbf{b}} - (\mathrm{RP}_{\mathbf{b}} - x) \cdot \delta_{\mathbf{b}}$ and $x_{\leftarrow\mathbf{s}} = \mathrm{RP}_{\mathbf{s}} + (x - \mathrm{RP}_{\mathbf{s}}) \cdot \delta_{\mathbf{s}}$. If a value x is backward propagated n times for agent $\mathbf{a}$, we write $x_{\leftarrow n[\mathbf{a}]}$, e.g. $x_{\leftarrow 2[\mathbf{a}]} = (x_{\leftarrow\mathbf{a}})_{\leftarrow\mathbf{a}}$. If a value is backward propagated for more than one agent, we list them left to right in the subscript, e.g., $x_{\leftarrow\mathbf{b}2[\mathbf{s}]} = ((x_{\leftarrow\mathbf{b}})_{\leftarrow\mathbf{s}})_{\leftarrow\mathbf{s}}$. The values of $x^*(t)$ can be calculated recursively from $t = T - 1$ back to $t = 0$ as follows:

$$x^*(t) = \begin{cases} \mathrm{RP}_{\iota(t+1)} & \text{if } t = T - 1 \\ (x^*(t+1))_{\leftarrow\iota(t+1)} & \text{if } t < T - 1 \end{cases}$$

It can be easily observed that $x_{\leftarrow \mathbf{b}} \geq x$ as $x_{\leftarrow \mathbf{b}} - x = \mathrm{RP}_{\mathbf{b}} - (\mathrm{RP}_{\mathbf{b}} - x) \cdot \delta_{\mathbf{b}} - x = (1 - \delta_{\mathbf{b}})(\mathrm{RP}_{\mathbf{b}} - x) \geq 0$, and $x_{\leftarrow \mathbf{s}} \leq x$ as $x_{\leftarrow \mathbf{s}} - x = \mathrm{RP}_{\mathbf{s}} + (x - \mathrm{RP}_{\mathbf{s}}) \cdot \delta_{\mathbf{s}} - x = (\delta_{\mathbf{s}} - 1)(x - \mathrm{RP}_{\mathbf{s}}) \leq 0$. In addition, if $x \leq \mathrm{RP}_{\mathbf{b}}$, it follows that $x_{\leftarrow \mathbf{b}} \leq \mathrm{RP}_{\mathbf{b}}$. Similarly, If $x \geq \mathrm{RP}_{\mathbf{s}}$, $x_{\leftarrow \mathbf{s}} \geq \mathrm{RP}_{\mathbf{s}}$.

Finally, agents' equilibrium strategies can be defined on the basis of $x^*(t)$ as follows:

$$\sigma^*_{\mathbf{b}}(t) = \begin{cases} t = 0 & \textit{offer}[x^*(0)] \\ 0 < t < T & \begin{cases} \text{if } \sigma_{\mathbf{s}}(t-1) = \textit{offer}[x] \text{ with } x \leq (x^*(t))_{\leftarrow \mathbf{b}} & \textit{accept} \\ \text{otherwise} & \textit{offer}[x^*(t)] \end{cases} \\ T \leq t \leq T_{\mathbf{b}} & \begin{cases} \text{if } \sigma_{\mathbf{s}}(t-1) = \textit{offer}[x] \text{ with } x \leq \mathrm{RP}_{\mathbf{b}} & \textit{accept} \\ \text{otherwise} & \textit{exit} \end{cases} \\ T_{\mathbf{b}} < t & \textit{exit} \end{cases}$$

$$\sigma^*_{\mathbf{s}}(t) = \begin{cases} t = 0 & \textit{offer}[x^*(0)] \\ 0 < t < T & \begin{cases} \text{if } \sigma_{\mathbf{b}}(t-1) = \textit{offer}[x] \text{ with } x \geq (x^*(t))_{\leftarrow \mathbf{s}} & \textit{accept} \\ \text{otherwise} & \textit{offer}[x^*(t)] \end{cases} \\ T \leq t \leq T_{\mathbf{s}} & \begin{cases} \text{if } \sigma_{\mathbf{b}}(t-1) = \textit{offer}[x] \text{ with } x \geq \mathrm{RP}_{\mathbf{s}} & \textit{accept} \\ \text{otherwise} & \textit{exit} \end{cases} \\ T_{\mathbf{s}} < t & \textit{exit} \end{cases}$$

Therefore, at equilibrium, the two agents will reach an agreement at the time $t = 1$ and the agreement price is $x^*(0)$. Agents' bargaining power depends on the order of proposing: the agent $\iota(T-1)$ that will act at the time point before the deadline has a stronger power, and the agent $\iota(T)$ gets a utility of 0.

3 One-to-Many Alternating-Offers Negotiation

3.1 Negotiation Mechanism

In this section, we extend the alternating-offers protocol to capture the situation wherein there is one buyer agent $\mathbf{b}$ and a set $\mathcal{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ of n seller agents such that: 1) the items sold by the sellers are the same, 2) all the sellers have exactly one item to sell, and 3) the buyer is interested in buying exactly one item.

Our mechanism extends the alternating-offers protocol allowing the buyer to carry on more simultaneous negotiations, each one with a different seller. As in [2, 19, 26], a buyer synchronously negotiates with multiple sellers in discrete time. We use the term "negotiation thread" for the single bargaining between $\mathbf{b}$ and a seller $\mathbf{s}_i$ and we denote it by $\Im_{\mathbf{b},\mathbf{s}_i}$. Furthermore, we denote by $\iota(\Im_{\mathbf{b},\mathbf{s}_i}, t)$ the agent that acts at t in the negotiation thread $\Im_{\mathbf{b},\mathbf{s}_i}$. We assume that if $\iota(\Im_{\mathbf{b},\mathbf{s}_i}, t) = \mathbf{b}$ then $\iota(\Im_{\mathbf{b},\mathbf{s}_j}, t) = \mathbf{b}$ for all j. That is, $\mathbf{b}$ simultaneously acts in all the negotiation threads. Therefore, if $\mathbf{b}$ is proposing at time t, $\iota(t) = \mathbf{b}$. Otherwise, $\iota(t) = \mathcal{S}$.

We modify the alternating-offers mechanism by introducing an action *confirm* to avoid agents' non-reasonable behaviors. In the following we show an example of non-reasonable behavior in absence of such action. The sellers' action space is $A = \{\textit{offer}[x], \textit{accept}, \textit{exit}, \textit{confirm}\}$, whereas the buyer's action space is the Cartesian product $\times_{i=1}^{n} A$. Legal actions for the buyer are all the pure strategies $\sigma_{\mathbf{b}} = \langle \sigma_{\mathbf{b},\mathbf{s}_1}, \ldots, \sigma_{\mathbf{b},\mathbf{s}_n} \rangle$ such that: if $\sigma_{\mathbf{s}_i}(t-1) \neq \textit{accept}$, then $\sigma_{\mathbf{b},\mathbf{s}_i}(t) \in \{\textit{offer}[x], \textit{accept}, \textit{exit}\}$ except when $t = 0$, *accept* is not available, otherwise $\sigma_{\mathbf{b},\mathbf{s}_i}(t) \in \{\textit{confirm}, \textit{exit}\}$. Legal actions for the sellers are defined analogously: if $\sigma_{\mathbf{b},\mathbf{s}_i}(t-1) \neq \textit{accept}$, then $\sigma_{\mathbf{s}_i}(t) \in \{\textit{offer}[x], \textit{accept}, \textit{exit}\}$ except when $t = 0$, *accept* is not available, otherwise $\sigma_{\mathbf{s}_i}(t) \in \{\textit{confirm}, \textit{exit}\}$. The action *confirm* is allowed only after making the action *accept*.

The outcome of a single negotiation thread $\Im_{\mathbf{b},\mathbf{s}_i}$ is $NoAgre\text{-}ement$ if either $\mathbf{b}$ or $\mathbf{s}_i$ made *exit*, whereas it is an agreement (x, t) if $\sigma_{\iota(\Im_{\mathbf{b},\mathbf{s}_i}, t)}(t) = \textit{confirm}$, where x is such that $\sigma_{\iota(\Im_{\mathbf{b},\mathbf{s}_i}, t-2)}(t-2) = \textit{offer}[x]$. Notice that, in absence of the action *confirm*, if $\mathbf{b}$ makes offers to more sellers and all these accept, $\mathbf{b}$ must buy more items. In presence of the action *confirm*, $\mathbf{b}$ is in the position to choose only one contract. Summarily, in our mechanism the following process is needed for implementing an agreement: one agent proposes a price, the other agent accepts the offer, then the first agent confirms the contract made by the second agent. Without loss of generality, we assume that each seller's deadline is no less than 2, i.e., $T_{\mathbf{s}_i} \geq 2$.

The utility functions of the seller agents are exactly those defined in the previous section. However, we need to refine the utility function of $\mathbf{b}$. This is because $\mathbf{b}$ can potentially buy more items, but is interested in only one item. We redefine $\mathbf{b}$'s utility as follows. If $\mathbf{b}$ has reached more than one agreement, let (x_{first}, t_{first}) be the agreement such that, for any other agreement (x_j, t_j), (1) $t_{first} \leq t_j$ and (2) $x_{first} \leq x_j$ if $t_{first} = t_j$. Let i_{first} be the seller involved in the agreement (x_{first}, t_{first}). Agent $\mathbf{b}$'s utility is defined over the set of agreements it reached:

$$U_{\mathbf{b}}(\{(x_i, t_i)\}) = \begin{cases} (\mathrm{RP}_{\mathbf{b}} - x_{first}) \cdot \delta_{\mathbf{b}}^{t_{first}} - \sum_{j \neq i_{first}} x_j & \text{if } t_{first} \leq T_{\mathbf{b}} \\ -\epsilon & \text{otherwise} \end{cases}$$

That is, $\mathbf{b}$ receives a positive utility from the first agreement, whereas all the other agreements reduce $\mathbf{b}$'s utility. This will induce a rational buyer to reach at most one agreement.

3.2 Agents' Equilibrium Strategies

Let $\mathcal{S}_{=t}$ be the set of sellers whose deadline is t, i.e., $\mathcal{S}_{=t} = \{\mathbf{s}_i | T_{\mathbf{s}_i} = t\}$. Let $\mathcal{S}_t$ be the set of sellers which have no shorter deadline than t, i.e., $\mathcal{S}_t = \{\mathbf{s}_i | T_{\mathbf{s}_i} \geq t\} = \cup_{t' \geq t} \mathcal{S}_{=t}$. Without loss of generality, we assume that the sellers $\mathcal{S}_t$ are ranked according to their reserve prices. We denote by $\mathcal{S}_t^i$ ($\mathcal{S}_{=t}^i$) the seller with the i^{th} lowest reserve price in $\mathcal{S}_t$ ($\mathcal{S}_{=t}$). Let $x^*_{\mathbf{b},\mathbf{s}_i}(t)$ be $\mathbf{b}$'s optimal offer to $\mathbf{s}_i$ at time t if $\iota(\Im_{\mathbf{b},\mathbf{s}_i}, t) = \mathbf{b}$ and $x^*_{\mathbf{s}_i,\mathbf{b}}(t)$ be $\mathbf{s}_i$'s optimal offer to agent $\mathbf{b}$ at time t if $\iota(\Im_{\mathbf{b},\mathbf{s}_i}, t) = \mathbf{s}_i$.

The negotiation deadline for the negotiation thread between $\mathbf{b}$ and $\mathbf{s}_i$ is $T_{\mathbf{b},\mathbf{s}_i} = \min(T_{\mathbf{b}}, T_{\mathbf{s}_i})$. After $T_{\mathbf{b},\mathbf{s}_i}$, at least one agent will have no interest in reaching agreements. Obviously, the negotiation deadline for $\mathbf{b}$ is $T = \max_{\mathbf{s}_i \in \mathcal{S}}\{T_{\mathbf{b},\mathbf{s}_i}\}$. We state the following lemma that allow us to reduce the complexity of the problem.

Lemma 1. *It is* $\mathbf{b}$*'s weakly dominant strategy to make the same offer to all the sellers in* $\mathcal{S}_{t+2}$ *at each time* t*.*

Proof. At t we consider only $\mathcal{S}_{t+2}$ since all the other sellers will not be interested in reaching agreements at $t+2$ and later. Consider the time point t wherein $\iota(t) = \mathbf{b}$. On the equilibrium path, at t agent $\mathbf{b}$ will expect to reach exactly one agreement, say $(x^*_{\mathbf{b}}(t+2), t+2)$, with a specific seller, say $\mathbf{s}^*$. Obviously, $\mathbf{s}^*$ is the seller that will accept the greatest offer. If $\mathbf{b}$ makes offers greater than $x^*_{\mathbf{b}}(t)$ to the other sellers, then these sellers will not accept such offers and therefore $\mathbf{b}$ cannot improve its utility. Analogously, if $\mathbf{b}$ makes offers lower than $x^*_{\mathbf{b}}(t)$ to the other sellers, it cannot improve its utility. □

According to Lemma 1 we can assume, without loss of generality, that $x^*_{\mathbf{b},\mathbf{s}_i}(t) = x^*_{\mathbf{b},\mathbf{s}_j}(t)$ for all i, j. For simplicity, we denote such offer by $x^*_{\mathbf{b}}(t)$. We state the following theorem whose proof is reported in Appendix A.

Theorem 2. *In the one-to-many negotiation, the sequences of equilibrium offers* $x^*_{\mathbf{b}}$ *and* $x^*_{\mathbf{s}_i}$ *are:*

$$x^*_{\mathbf{b}}(t) = \begin{cases} \mathrm{RP}_{\mathcal{S}^1_{t+2}} & t = T-2 \text{ or } t = T_{\mathcal{S}^1_{t+2}} - 2 \\ \min\{(x^*_{\mathcal{S}^1_{t+2}}(t+1))_{\leftarrow \mathcal{S}^1_{t+2}}, \mathrm{RP}_{\mathcal{S}^2_{t+2}}\} & t < T-2 \text{ and } t \neq T_{\mathcal{S}^1_{t+2}} - 2 \end{cases},$$

$$x^*_{\mathbf{s}_i}(t) = \begin{cases} \max\{\mathrm{RP}_{\mathbf{s}_i}, \mathrm{RP}_{\mathcal{S}^2_T}\} & t = T-2 \\ \max\{\mathrm{RP}_{\mathbf{s}_i}, \min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}}\}\} & t < T-2 \end{cases}.$$

Agent's equilibrium strategies are similar to those discussed in Section 2, but $\sigma_{\mathbf{b},\mathbf{s}_i}$ *prescribes that:*

- $\mathbf{b}$ *accepts the offer* x *made by* $\mathbf{s}_i$ *at* t *if:* $x \leq (x^*_{\mathbf{b}}(t))_{\leftarrow \mathbf{b}}$ *and* x *is the lowest received offer. If more than one seller has offered* x*, than* $\mathbf{b}$ *accepts the offer made by the seller with the lowest reserve price;*

- $\mathbf{b}$ *confirms an accept of* $\mathbf{s}_i$ *at* t *if:* $\sigma_{\mathbf{b}}(t-2) = \mathit{offer}[x]$ *with* $x \leq (x^*_{\mathbf{b}}(t))_{\leftarrow 2[\mathbf{b}]}$ *and, among all the sellers that have accepted* $\sigma_{\mathbf{b}}(t-2)$*,* $\mathbf{s}_i$ *is the one with the lowest reserve price;*

and $\sigma_{\mathbf{b},\mathbf{s}_i}$ *prescribes that:*

- $\mathbf{s}_i$ *confirms the accept of* $\mathbf{b}$ *at* t *if:* $\sigma_{\mathbf{s}_i}(t-2) = \mathit{offer}[x]$ *with* $x \geq \max\{(x^*_{\mathbf{s}_i}(t))_{\leftarrow 2[\mathbf{s}_i]}, \mathrm{RP}_{\mathbf{s}_i}\}$.

The computational complexity of the backward induction is $\mathcal{O}(nT)$ as the backward induction will go through all the time points and at each time point, each agent has at most three possible optimal actions. The equilibrium agreement is reached at $t = 2$ between $\mathbf{b}$ and $\mathcal{S}^1_2$ and it is $(x^*_{\mathbf{b}}(0), 2)$ if $\iota(0) = \mathbf{b}$ and $(x^*_{\mathcal{S}^1_2}(0), 2)$ otherwise. It can be easily observed that $\mathrm{RP}_{\mathcal{S}^1_2} \leq x^*_{\mathbf{b}}(0), x^*_{\mathcal{S}^1_2}(0) \leq \mathrm{RP}_{\mathcal{S}^2_2}$. The result about agreement price is intuitive in the following sense: obviously, the agreement price cannot be lower than each seller's reserve price. But it also cannot be higher than the second lowest price as, if so, there is at least another seller who is willing to sell for less and make an agreement with the buyer. Therefore, market competition guarantees that the buyer can make an agreement by paying no more than $\mathrm{RP}_{\mathcal{S}^2_2}$. The lower bound of agreement is due to the proposing ordering. For example, if $T = 2$ and the buyer proposes at time $t = 0$, the buyer will propose $\mathrm{RP}_{\mathcal{S}^1_2}$ and the agent $\mathcal{S}^1_2$ will accept the offer at time $t = 1$. We can see that the market competition plays an important role in affecting negotiation results. The buyer can make an agreement with price at most $\mathrm{RP}_{\mathcal{S}^2_2}$. With more sellers, the buyer can get better (at least not worse) negotiation result.

Let us remark an observation. Consider the situation wherein $\iota(0) = \mathcal{S}$ and $x^*_{\mathcal{S}^1_2} = \mathrm{RP}_{\mathcal{S}^2_2}$. Although both $\mathcal{S}^1_2$ and $\mathcal{S}^2_2$ have the same equilibrium offer, i.e., $\mathrm{RP}_{\mathcal{S}^2_2}$, the equilibrium strategy of $\mathbf{b}$ prescribes that $\mathbf{b}$ must accept only the offer made by $\mathcal{S}^1_2$. In the case $\mathbf{b}$ accepts the offer by $\mathcal{S}^2_2$ or randomizes over accepting those offers, $\mathcal{S}^1_2$'s optimal action at $t = 0$ does not exist, being $\lim_{\varepsilon \to 0}(\mathcal{S}^2_2 - \varepsilon)$ with $\varepsilon \neq 0$. We can state the following theorem which is a direct consequence of the above observation and of the equilibrium uniqueness in bilateral alternating-offers.

Theorem 3. *Agents' strategies on the equilibrium path are unique except when* $\mathrm{RP}_{\mathcal{S}^1_2} = \mathrm{RP}_{\mathbf{s}_i}$ *for more than one* i*.*

Notice that, when the reserve price of more sellers is equal to $\mathrm{RP}_{\mathcal{S}^1_2}$, all these sellers will offer their reserve price and $\mathbf{b}$ can accept any single offer among these. However, it can be easily observed that all the equilibria are equivalent in terms of agents' payoffs, $\mathbf{b}$ receiving the same utility in all the equilibria. As we assume that agents have different reserve prices, the equilibrium is unique.

Fig. 1 shows an example of backward induction construction with $\mathrm{RP}_{\mathbf{b}} = 1$, $\mathrm{RP}_{\mathbf{s}_1} = 0$, $\mathrm{RP}_{\mathbf{s}_2} = 0.2$, $\delta_{\mathbf{b}} = 0.8$, $\delta_{\mathbf{s}_1} = 0.7$, $\delta_{\mathbf{s}_2} = 0.8$, $T_{\mathbf{b}} = 10$, $T_{\mathbf{s}_1} = 11$, $T_{\mathbf{s}_2} = 7$. We report in the figure for any time point t the optimal offer $x^*_{\mathbf{a}}(t)$ that $\iota(t)$ can make; the dashed lines are sellers' optimal offers if there is only one seller. The time point from which we can apply the backward induction method is $T = 10$ at which $\mathbf{b}$ will confirm the agreement made at $t = 9$. At $t = 9$ agent $\mathbf{s}_1$ will accept any offer equal to or greater than its reserve price $\mathrm{RP}_{\mathbf{s}_1} = 0$. The optimal offer $x^*_{\mathbf{b}}(8)$ of $\mathbf{b}$ at $t = 8$ is thus $\mathrm{RP}_{\mathbf{s}_1} = 0$. $\mathbf{s}_1$'s optimal offer $x^*_{\mathbf{s}_1}(7)$ at $t = 7$ is $(x^*_{\mathbf{b}}(8))_{\leftarrow \mathbf{b}} = \mathrm{RP}_{\mathbf{b}} - (\mathrm{RP}_{\mathbf{b}} - x^*_{\mathbf{b}}(8))\delta_{\mathbf{b}} = 0.2$. $\mathbf{b}$'s optimal offer at time $t = 6$ is then $x^*_{\mathbf{b}}(6) = (x^*_{\mathbf{s}_1}(7))_{\leftarrow \mathbf{s}_1} = 0.14$. At time $t = 5$, another seller $\mathbf{s}_2$ can make an offer (note that $t = 5$ is the last time $\mathbf{s}_2$ can make an offer as it needs another two rounds to accept and confirm an agreement). $\mathbf{s}_1$ and $\mathbf{s}_2$ will compete with each other and their optimal offers aren't $(x^*_{\mathbf{b}}(6))_{\leftarrow \mathbf{b}} = 0.312$ as one seller has an incentive to choose a lower price if the other seller choose $(x^*_{\mathbf{b}}(6))_{\leftarrow \mathbf{b}} = 0.312$. The equilibrium optimal price for the two sellers is $x^*_{\mathbf{s}_1}(5) = x^*_{\mathbf{s}_2}(5) = \mathrm{RP}_{\mathcal{S}^2_{t=5+2}} = \mathrm{RP}_{\mathbf{s}_2} = 0.2$. The process continues to the initial time point $t = 0$ where $\mathbf{b}$'s optimal offer is $x^*_{\mathbf{b}}(0) = 0.14$.

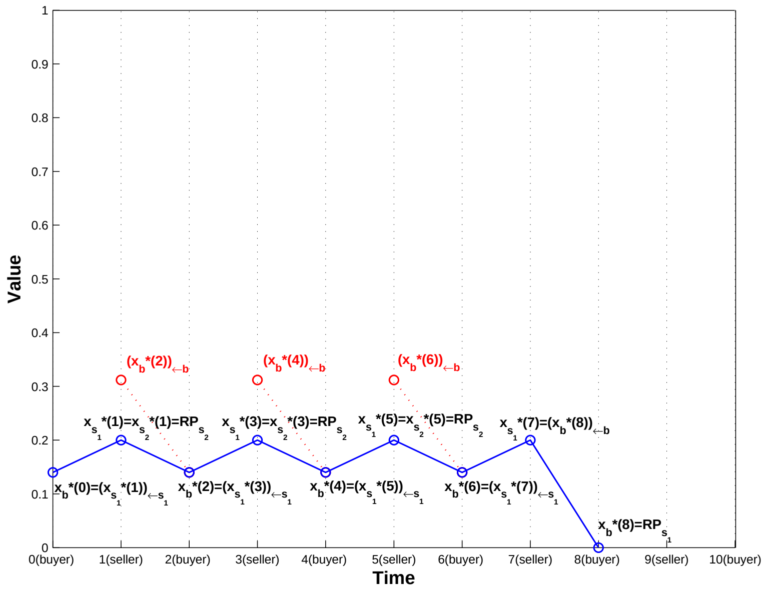

Figure 1: Backward induction construction with $\mathrm{RP}_{\mathbf{b}} = 1$, $\mathrm{RP}_{\mathbf{s}_1} = 0$, $\mathrm{RP}_{\mathbf{s}_2} = 0.2$, $\delta_{\mathbf{b}} = 0.8$, $\delta_{\mathbf{s}_1} = 0.7$, $\delta_{\mathbf{s}_2} = 0.8$, $T_{\mathbf{b}} = 10$, $T_{\mathbf{s}_1} = 11$, $T_{\mathbf{s}_2} = 7$; at each time point t the optimal offer $x^*_{\mathbf{a}}(t)$ that $\iota(t)$ can make is marked; the dashed lines are sellers' optimal offer if there is only one seller.

There are some other mechanisms which can be used to implement contracts between buyer $\mathbf{b}$ and sellers $\mathcal{S}$. Here we compare our model with the following mechanisms:

- *Bilateral bargaining without outside option*: Rubinstein's bilateral bargaining does not offer any mechanism to capture competition between sellers. In order to compare outcomes from bilateral bargaining with respect to outcomes from our mechanism, suppose that $\mathbf{b}$ is able to choose the seller with which to negotiate. In our mechanism the buyer $\mathbf{b}$ gains as in bilateral bargaining without outside option when the sequence of optimal offers $x^*(t)$ in the bilateral negotiation between $\mathbf{b}$ and $\mathcal{S}_2^1$ is such that $x_i^*(t) \leq \mathrm{RP}_{\mathcal{S}_2^2}$, otherwise the buyer $\mathbf{b}$ gains more in our mechanism.

- *Bilateral bargaining with outside option*: In our mechanism the buyer gains no less than in bilateral bargaining with outside option in which an agent can leave the bilateral negotiation it is currently carrying on and negotiate with a different opponent [4]. We report an example. Consider the situation where there are two sellers with the same reservation price $\mathrm{RP}_{\mathbf{s}}$ and any deadline. In bilateral bargaining with outside option the agreement price is strictly larger than $\mathrm{RP}_{\mathbf{s}}$, instead in our protocol the agreement price is exactly $\mathrm{RP}_{\mathbf{s}}$.

- *VCG auction*: Since VCG auction does not take into account any temporal issues (no deadline and no discount factor), we limit our comparison to the agreement price. In VCG mechanism the agreement price is exactly $\mathrm{RP}_{\mathcal{S}_2^2}$, whereas in our bargaining model the buyer's agreement price falls between $[\mathrm{RP}_{\mathcal{S}_2^1}, \mathrm{RP}_{\mathcal{S}_2^2}]$.

4 Many-to-Many Alternating-Offers Negotiation

4.1 Negotiation Mechanism

In this section, we propose a bargaining model for many-to-many negotiation where m buyer agents $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ negotiate n seller agents $\mathcal{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$. In this case, both buyers and sellers face competition and multiple contracting opportunities. Again, we assume that the items sold by the sellers or bought by buyers are equal, and each agent has only one item to buy or sell.

In the many-to-many negotiation case, each agent concurrently negotiates with many trading partners. Agent $\mathbf{b}_j$'s concurrent negotiation includes at most n threads $\Im_{\mathbf{b}_j,\mathcal{S}} = \{\Im_{\mathbf{b}_j,\mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}\}$, where $\Im_{\mathbf{b}_j,\mathbf{s}_i}$ represents the negotiation thread between $\mathbf{b}_j$ and seller $\mathbf{s}_i$. We still assume that, at each time, either the buyers propose to all the sellers ($\iota(t) = \mathcal{B}$) or the sellers propose to all the buyers ($\iota(t) = \mathcal{S}$). Similarly, let $\mathcal{B}_{=t}$ be the set of buyers whose deadline is t, i.e., $\mathcal{B}_{=t} = \{\mathbf{b}_j | T_{\mathbf{b}_j} = t\}$. Let $\mathcal{B}_t$ be the set of buyers which have no shorter deadline than t and $\mathcal{B}_t^i$ ($\mathcal{B}_{=t}^i$) is the buyer with the i^{th} highest reserve price in $\mathcal{B}_t$ ($\mathcal{B}_{=t}$).

We still use action *confirm* to avoid one agent's making more than one final agreement. Buyers and sellers' action space and agents' legal actions at each time are the same as that in one-to-many negotiation. The utility functions of the buyer agents are exactly those defined in the previous section. However, we need to refine the utility function of $\mathbf{s}_i$ as it can potentially sell more items, but it has only one item to sell. We redefine $\mathbf{s}_i$'s utility as follows. If $\mathbf{s}_i$ has reached more than one final agreement, it gets a utility of $-\infty$. Otherwise, it's utility is the same as that in bilateral negotiation. Therefore, $\mathbf{s}_i$ will make at most one final agreement.

4.2 Agents' Equilibrium Strategies

The negotiation deadline for the negotiation between agent $\mathbf{b}_j$ and seller $\mathbf{s}_i$ is $T_{\mathbf{b}_j,\mathbf{s}_i} = \min(T_{\mathbf{b}_j}, T_{\mathbf{s}_i})$. The negotiation deadline for the agent $\mathbf{b}_j$ is $T_{\mathbf{b}_j,\mathcal{S}} = \max_{\mathbf{s}_i \in \mathcal{S}} T_{\mathbf{b}_j,\mathbf{s}_i}$. Let $x^*_{\mathbf{b}_j,\mathbf{s}_i}(t)$ be $\mathbf{b}_j$'s optimal offer to agent $\mathbf{s}_i$ at t if $\iota(t) = \mathcal{B}$ and $x^*_{\mathbf{s}_i,\mathbf{b}_j}(t)$ be $\mathbf{s}_i$'s optimal offer to agent $\mathbf{b}_j$ at time t if $\iota(t) = \mathcal{S}$.

Lemma 4. *It is each agent's dominant strategy to propose the same price to all the trading partners at each time t.*

Proof. The proof is the same as the proof of Lemma 1. □

Then we use $x^*_{\mathbf{b}_j}(t)$ for short to represent $\mathbf{b}_j$'s optimal offer at t if $\iota(t) = \mathcal{B}$ and use $x^*_{\mathbf{s}_i}(t)$ to represent $\mathbf{s}_i$'s optimal offer at time t if $\iota(t) = \mathcal{S}$.

Lemma 5. *In equilibrium, agents of the same type should have the same equilibrium winning price (a price acceptable to agents of the different type).*

Proof. Let's prove this by contradiction. Assume two buyers have different winning prices at some time t, i.e., the lowest price acceptable to any seller. Then the seller who is willing to accept the the lower winning price should change to accept the higher winning price. Therefore, the two winning prices are not in equilibrium. □

We state the following theorem whose proof is in Appendix B.

Theorem 6. *In the many-to-many negotiation, the sequences of optimal offers in equilibrium are: Buyer $\mathbf{b}_j$'s optimal offer at time $t \leq T_{\mathbf{b}_j} - 2$ is $x^*_{\mathbf{b}_j}(t) = \min(\mathrm{RP}_{\mathbf{b}_j}, x^*_{\mathcal{B}}(t))$. Seller $\mathbf{s}_i$'s optimal offer at $t \leq T_{\mathbf{s}_i} - 2$ is $x^*_{\mathbf{s}_i}(t) = \max(\mathrm{RP}_{\mathbf{s}_i}, x^*_{\mathcal{S}}(t))$.*

$x^*_{\mathcal{B}}(t)$ is given by: 1) At $t = T - 2$, $x^*_{\mathcal{B}}(t) = \mathrm{RP}_{\mathcal{S}_{t+2}^{|\mathcal{B}_{t+2}|}}$ if $|\mathcal{B}_{t+2}| \leq |\mathcal{S}_{t+2}|$; otherwise, $x^*_{\mathcal{B}}(t) = \mathrm{RP}_{\mathcal{B}_{t+2}^{|\mathcal{S}_{t+2}|+1}}$. 2) At $t < T - 2$, $x^*_{\mathcal{B}}(t) = \max\{\mathrm{RP}_{\mathcal{B}_{t+2}^{|\mathcal{S}_{t+2}|+1}}, \{\{(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{s}_i} | \mathbf{s}_i \in$

$\mathcal{S}_{t+2} - \mathcal{S}_{t+3}\}\big\}_{|\mathcal{S}_{t+2}|}\big\}$ if $|\mathcal{S}_{t+2}| < |\mathcal{B}_{t+2}|$. Otherwise, $x^*_{\mathcal{B}}(t) = \big\{\{(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+2} - \mathcal{S}_{t+3}\}\big\}_{|\mathcal{B}_{t+2}|}$. In the above equations, $\mathcal{Y}_i$ ($\mathcal{Y}^i$) denotes the i^{th} smallest (largest) value in the value set $\mathcal{Y}$.

$x^*_{\mathcal{S}}(t)$ is given by: 1) At $t = T-2$, $x^*_{\mathcal{S}}(t) = \mathrm{RP}_{\mathcal{B}_T^{|\mathcal{S}_{t+2}|}}$ if $|\mathcal{S}_T| \le |\mathcal{B}_T|$, $x^*_{\mathcal{S}}(t) = \mathrm{RP}_{\mathcal{S}_{t+2}^{|\mathcal{B}_{t+2}|+1}}$ if $|\mathcal{S}_{t+2}| > |\mathcal{B}_{t+2}|$. 2) At $t < T-2$, $x^*_{\mathcal{S}}(t) = \big\{\{(x^*_{\mathbf{b}_j}(t+1))_{\leftarrow \mathbf{b}_j} | \mathbf{b}_j \in \mathcal{B}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{b}_j} | \mathbf{b}_j \in \mathcal{B}_{=t+2}\}\big\}^{|\mathcal{S}_{t+2}|}$ if $|\mathcal{S}_{t+2}| \le |\mathcal{B}_{t+2}|$. Otherwise, $x^*_{\mathcal{S}}(t) = \min\big\{\mathrm{RP}_{\mathcal{S}_{t+2}^{|\mathcal{B}_{t+2}|+1}}, \big\{\{(x^*_{\mathbf{b}_j}(t+1))_{\leftarrow \mathbf{b}_j} | \mathbf{b}_j \in \mathcal{B}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{b}_j} | \mathbf{b}_j \in \mathcal{B}_{=t+2}\}\big\}^{|\mathcal{B}_{t+2}|}\big\}$.

Based on $x^*_{\mathbf{b}_j}(t)$ and $x^*_{\mathbf{s}_i}(t)$, we can get agents' optimal actions in the same way as that in Theorem 2 except that an agent needs to use the following rule while accepting offers or confirming accepts: a buyer $\mathbf{b}_j$ accepts the offer x made by $\mathbf{s}_i$ at t if: $x \le (x^*_{\mathbf{b}_j}(t))_{\leftarrow \mathbf{b}_j}$ and x is the lowest received offer. If more than one seller has offered x and buyer $\mathbf{b}_j$ has the q^{th} highest reserve price in $\mathcal{B}_{t+2}$, $\mathbf{b}_j$ accepts the offer made by the seller with the q^{th} lowest reserve price in sellers $\mathcal{S}_{t+2}$.[1] Similarly, if buyer $\mathbf{b}_j$ intends to confirm an agreement with price x and multiple sellers have made the same agreement, $\mathbf{b}_j$ will confirm the agreement made by the seller with the q^{th} lowest reserve price in sellers $\mathcal{S}_{t+2}$. To save space, the details of sellers' optimal actions are omitted here.

The computational complexity of the backward induction is $\mathcal{O}((n+m)T)$ as the backward induction will go through all the time points and at each time point, each agent has at most $n+m$ possible optimal actions. It is easy to see that the bargaining agreement in the many-to-many negotiation is $\big(x^*_{\mathcal{B}}(0), 2\big)$ if $\iota(0) = \mathcal{B}$ and is $\big(x^*_{\mathcal{S}}(0), 2\big)$ if $\iota(0) = \mathcal{S}$. In addition, when the number of buyers is not equal to the number of sellers, the market competition affects the equilibrium price in the following way: if the number of buyers is less than the number of sellers, the buyers have larger bargaining power which increases with the number of sellers and decreases with the number of buyers. In contrast, if the number of buyers is larger than the number of sellers, the buyers have less bargaining power. The proposing order also affects the equilibrium price.

Fig. 2 shows an example of backward induction construction in many-to-many negotiation. The setting in Fig. 2 is the same as that in Fig. 1 except that there is another buyer $\mathbf{b}'$ with parameters $\mathrm{RP}_{\mathbf{b}'} = 0.9$, $\delta_{\mathbf{b}'} = 0.7$, and $T_{\mathbf{b}'} = 6$. We report in the figure for any time t the optimal offer $x^*_{\mathcal{B}}(t)$ or $x^*_{\mathcal{S}}(t)$. At time $t = 4$, $\mathbf{b}'$ can make an offer to compete with buyer $\mathbf{b}$. Thus we have $x^*_{\mathcal{B}}(4) = \{(x^*_{\mathbf{s}_1}(5))_{\leftarrow \mathbf{s}_1}, (x^*_{\mathbf{s}_2}(5))_{\leftarrow \mathbf{s}_2}\}_2 = \{0.14, 0.2\}_2 = 0.2$. The process continues to the initial time point $t = 0$ where $x^*_{\mathcal{B}}(0) = 0.40992$.

While there is two-sided competition in the market, market mechanisms like double auction can be used for resource allocation. The double auction is one of the most common exchange institutions where both sellers and buyers submit bids which are then ranked highest to lowest to generate demand and supply profiles. Double auctions permit multiple buyers and sellers to bid to exchange a designated commodity. Some double auction mechanisms (e.g., BBDA [8]) have been applied to trading in markets. A market mechanism is *efficient* if the goods are transferred to agents that value them most.

Theorem 7. *The many-to-many negotiation is efficient.*

Proof. This result is straightforward. Assume there are sellers $\mathbf{s}_i$ and $\mathbf{s}_j$ such that $\mathrm{RP}_{\mathbf{s}_i} > \mathrm{RP}_{\mathbf{s}_j}$. It is impossible that seller $\mathbf{s}_i$ makes an agreement but seller $\mathbf{s}_j$ fails as seller $\mathbf{s}_j$ can make an offer less than $\mathrm{RP}_{\mathbf{s}_i}$ and thus gains a contract with positive revenue. □

In a market consisting of two sets of agents, matching algorithms can also be used to solve agents' conflicts of resource requirements. Then we require a matching to be *stable*, i.e., it left no pair of agents on opposite sides of the market who were not matched to each other but would both prefer to be. Many-to-many negotiation

[1] Note that in equilibrium, when a buyer $\mathbf{b}_j$ with q^{th} highest reserve price is accepting an offer with price x, the number of sellers proposing x at $t-1$ should be no less than q. The proof is omitted as it can be easily derived from the process of calculating agents' optimal prices.

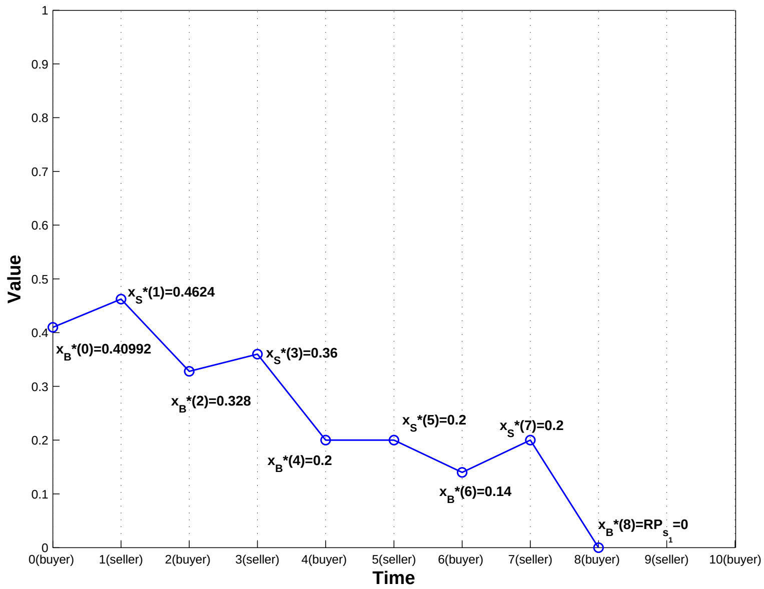

Figure 2: Backward induction construction. At each time t the optimal offer $x^*_{\mathcal{B}}(t)$ or $x^*_{\mathcal{S}}(t)$ is marked.

allows one to avoid studying matching mechanisms since each agent is implicitly matched with all its trading partners.

5 Uncertainty about Reserve Prices

In this section we analyze the impact of uncertain information about agents' reserve prices. We focus on the situation that one buyer $\mathbf{b}$ is negotiating with a number of sellers $\mathcal{S}$ and there is one-sided two-type uncertainty about the buyer's reserve price. We choose to study two-type uncertainty about the buyer's reserve price in one-to-many negotiation for a number of reasons. First, there is no applicable algorithm for computing sequential equilibria in bargaining with uncertain reserve prices, even for bilateral bargaining in which each agent has a continuous strategy space. Operational research inspired algorithms such as Miltersen-Sorensen [17] work only on games with finite number of strategies, and therefore cannot be applied to bargaining. Several attempts to extend the backward induction method [10] have been tried, but they work for very restrictive cases. This is because in the computation of the equilibrium they break down the circularity between strategies and belief system. The *only* known result about bargaining with uncertain reserve prices is due to Chatterjee and Samuelson [6,7] where they studied bilateral infinite horizon bargaining with two-type uncertainty over the reservation values. The absence of agents' deadlines makes these two results non applicable to the situation we study in the paper. Second, Gatti *et al*. [11] analyzed bilateral bargaining with one-sided uncertain deadlines. However, their approach cannot be applied to bargaining with uncertain reserve prices. Uncertainty with reserve prices is similar to uncertainty with discounting factors. Our solution can be applied to bargaining with uncertain discounting factors. Lastly, since it's very complicated to compute sequential equilibria in bargaining with more than two types of uncertainties or two-sided uncertainties, we consider one-sided uncertainty as the starting point. For the same reason, we consider two-type uncertainty in one-to-many negotiation. However, we discuss how to extend our analysis to many-to-many negotiation and multi-type uncertainty at the end of this section.

5.1 Introducing Uncertainty

With uncertain information, the appropriate solution concept for an extensive-form game is Kreps and Wilson's *sequential equilibrium* [16]. A sequential equilibrium is a pair $a = \langle \mu, \sigma \rangle$ (also called an assessment) where μ is a belief system that specifies how agents' beliefs evolve during the game and σ specifies agents' strategies. At an equilibrium μ must be *consistent* with respect to σ and σ must be *sequentially rational* given μ.

In this section we assume that the buyer $\mathbf{b}$ can be of two types: buyer $\mathbf{b}_h$ with a reserve price RP_h and buyer $\mathbf{b}_l$ with a reserve price RP_l such that $\mathrm{RP}_h > \mathrm{RP}_l$. We assume that each seller $\mathbf{s}$ has the initial belief about the type of the buyer $\mathbf{b}$. Thus, the initial belief of any seller $\mathbf{s}$ on $\mathbf{b}$ is $\mu(0) = \langle \Delta^0_{\mathbf{b}}, P^0_{\mathbf{b}} \rangle$ where $\Delta^0_{\mathbf{b}} = \{\mathbf{b}_h, \mathbf{b}_l\}$ and $P^0_{\mathbf{b}} = \{\omega^0_{\mathbf{b}_h}, \omega^0_{\mathbf{b}_l}\}$ where $\omega^0_{\mathbf{b}_h}$ ($\omega^0_{\mathbf{b}_l}$, respectively) is the *priori* probability that $\mathbf{b}$ is of type $\mathbf{b}_h$ ($\mathbf{b}_l$, respectively). It follows that $\omega^0_{\mathbf{b}_h} + \omega^0_{\mathbf{b}_l} = 1$. During bargaining, seller $\mathbf{s}$'s belief will evolve using the Bayes rule. It's easy to see that in incomplete information bargaining, it's still a weekly dominant strategy for the buyer $\mathbf{b}$ to make the same offer to all the sellers. Therefore, different sellers' beliefs about the type of buyer $\mathbf{b}$ will always be the same at any time t. The belief of $\mathbf{s}$ on the type of $\mathbf{b}$ at time t is $\mu(t)$. The probability assigned by $\mathbf{s}$ to $\mathbf{b} = \mathbf{b}_h$ at time t is denoted $\omega^t_{\mathbf{b}_h}$; the probability assigned to $\mathbf{b} = \mathbf{b}_l$ is $\omega^t_{\mathbf{b}_l} = 1 - \omega^t_{\mathbf{b}_h}$. Given an assessment $a = \langle \mu, \sigma \rangle$, there are two possible bargaining outcomes: outcome $o_{\mathbf{b}_h}$ if $\mathbf{b} = \mathbf{b}_h$ and $o_{\mathbf{b}_l}$ if $\mathbf{b} = \mathbf{b}_l$. We denote bargaining outcome as $o = \langle o_{\mathbf{b}_h}, o_{\mathbf{b}_l} \rangle$.

With pure strategies, buyer types' possible behaviors regarding whether they behave in the same way on the equilibrium path at each decision making node are finite. We use the term "choice rule" to characterize agents' strategies regarding whether they behave in the same way at a specific decision making point. Easily, at a decision making node $\mathbf{b}_l$ and $\mathbf{b}_h$ can make the same offer (in this case, choice rules are said *pooling*) or can make different offers (in this case, choice rules are said *separating*). On the basis of this consideration, we can make some assumptions over the belief system without loosing generality. On the equilibrium path $\mu(t) = \langle \Delta^t_{\mathbf{b}}, P^t_{\mathbf{b}} \rangle$ of $\mathbf{s}$ on $\mathbf{b}$ at any time t is one the following. After a time point t where buyer types' choice rule is pooling, $\mu(t+1) = \mu(t)$, i.e., $\Delta^{t+1}_{\mathbf{b}} = \Delta^t_{\mathbf{b}}$ and $P^{t+1}_{\mathbf{b}} = P^t_{\mathbf{b}}$. After a time point t where buyer types' choice rule is separating, there could be two possible beliefs: if the equilibrium offer of $\mathbf{b}_h$ has been observed, then $\Delta^{t+1}_{\mathbf{b}} = \{\mathbf{b}_h\}$ ($\mathbf{s}$ believes $\mathbf{b} = \mathbf{b}_h$ with certainty), which implies $\omega^{t+1}_{\mathbf{b}_h} = 1$ and $\omega^{t+1}_{\mathbf{b}_l} = 0$; if instead the equilibrium offer of $\mathbf{b}_l$ has been observed, $\Delta^{t+1}_{\mathbf{b}} = \{\mathbf{b}_l\}$ ($\mathbf{s}$ believes $\mathbf{b} = \mathbf{b}_l$ with certainty), which implies $\omega^t_{\mathbf{b}_h} = 0$ and $\omega^{t+1}_{\mathbf{b}_l} = 1$. We need also specify the belief system off the equilibrium path, i.e., when an agent makes an action that is not optimal. We use the *optimistic conjectures* [23]. That is, when $\mathbf{b}$ acts off the equilibrium strategy, agent $\mathbf{s}$ will believe that agent $\mathbf{b}$ is of its "weakest" type, i.e., of the type that would gain the least in a complete information game. This choice is directed to assure the existence of the equilibrium for the largest subset of the space of the parameters. In our case, the weakest type is $\mathbf{b}_l$ (we prove it in the following section). We can therefore specify $\mu(t)$ by specifying $\Delta^t_{\mathbf{b}}$. We will write $\mu(t) = \{\mathbf{b}_h, \mathbf{b}_l\}$, or $\mu(t) = \{\mathbf{b}_h\}$, or $\mu(t) = \{\mathbf{b}_l\}$.

5.2 Off the Equilibrium Path Optimal Strategies

Before analyzing equilibrium strategies when the buyer can be of two types, we provide the optimal strategies in the situations $\mathbf{s}$ believes the buyer of one single type. There are two cases: 1) Seller $\mathbf{s}$ has the right belief about the type of the buyer $\mathbf{b}$. In this case, agents' equilibrium strategies are the equilibrium strategies of the corresponding complete information bargaining discussed in Section 3. Let $x^c_{\mathbf{b}_h}(t)$ ($x^c_{\mathbf{b}_l}(t)$, respectively) be agents' optimal offer at time t when $\mathbf{b}$ is of type $\mathbf{b}_h$ ($\mathbf{b}_l$, respectively) in this case. That is, if $\iota(t) = \mathbf{b}$, $x^c_{\mathbf{b}_h}(t)$ is $\mathbf{b}$'s optimal offer $x^*_{\mathbf{b}}(t)$ at time t in complete information bargaining when it is of type $\mathbf{b}_h$. If $\iota(t) = \mathcal{S}$, $x^c_{\mathbf{b}_h}(t)$ is $\mathcal{S}$'s lowest optimal offer $x^*_{\mathcal{S}}(t)$ at t in complete information bargaining when $\mathbf{b}$ is of type $\mathbf{b}_h$. 2) Seller $\mathbf{s}$ has the wrong belief about the type of the buyer $\mathbf{b}$, i.e., $\mathbf{b}_h$ is believed to be $\mathbf{b}_l$ and $\mathbf{b}_l$ is believed to be $\mathbf{b}_h$.

Lemma 8. $x^c_{\mathbf{b}_h}(t) \geq x^c_{\mathbf{b}_l}(t)$.

Proof. We can proof the results from the proof of Theorem 2:

Case 1 ($\iota(T) = \mathbf{s}$). It follows that $x^c_{\mathbf{b}_h}(T-2) = x^c_{\mathbf{b}_l}(T-2) = \mathrm{RP}_{\mathcal{S}^2_T}$. Then we have $x^c_{\mathbf{b}_h}(T-3) = \min\{(x^c_{\mathbf{b}_h}(T-2))_{\leftarrow \mathcal{S}^1_{T-1}}, \mathrm{RP}_{\mathcal{S}^2_{T-1}}\} = \min\{(x^c_{\mathbf{b}_l}(T-2))_{\leftarrow \mathcal{S}^1_{T-1}}, \mathrm{RP}_{\mathcal{S}^2_{T-1}}\} = x^c_{\mathbf{b}_l}(T-3)$. At time $t = T-4$, we have $x^c_{\mathbf{b}_h}(t) = \min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^c_{\mathbf{b}_h}(t+1))_{\leftarrow \mathbf{b}}\} = \min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, \mathrm{RP}_h(1-\delta_{\mathbf{b}}) + \delta_{\mathbf{b}} x^c_{\mathbf{b}_h}(t+1)\} \geq \min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, \mathrm{RP}_h(1-\delta_{\mathbf{b}}) + \delta_{\mathbf{b}} x^c_{\mathbf{b}_l}(t+1) = x^c_{\mathbf{b}_l}(t)\}$. Recursively, we have $x^c_{\mathbf{b}_h}(t) \geq x^c_{\mathbf{b}_l}(t)$ for $t < T-4$.

Case 2 ($\iota(T) = \mathbf{b}$). It follows that $x^c_{\mathbf{b}_h}(T-2) = \mathrm{RP}_{\mathcal{S}^1_T} = x^c_{\mathbf{b}_l}(T-2)$. Then at time $T-3$, we have $x^c_{\mathbf{b}_h}(T-3) = \min\{\mathrm{RP}_{\mathcal{S}^2_{T-1}}, (x^c_{\mathbf{b}_h}(T-2))_{\leftarrow \mathbf{b}}\} = \min\{\mathrm{RP}_{\mathcal{S}^2_{T-1}}, \mathrm{RP}_h(1-\delta_{\mathbf{b}}) + \delta_{\mathbf{b}} x^c_{\mathbf{b}_h}(T-2)\} \geq \min\{\mathrm{RP}_{\mathcal{S}^2_{T-1}}, \mathrm{RP}_h(1-\delta_{\mathbf{b}}) + \delta_{\mathbf{b}} x^c_{\mathbf{b}_l}(T-2) = x^c_{\mathbf{b}_l}(T-3)\}$. Recursively, we have $x^c_{\mathbf{b}_h}(t) \geq x^c_{\mathbf{b}_l}(t)$ for $t < T-3$. □

We can see that $\mathbf{b}_h$ is weaker than $\mathbf{b}_l$ in terms of its offering price at each time point in complete information bargaining.

Lemma 9. $\mathrm{RP}_h - x^c_{\mathbf{b}_h}(t) \geq \mathrm{RP}_l - x^c_{\mathbf{b}_l}(t)$.

Proof. We can get this result by following the same procedure in the proof of Lemma 8. □

$\mathrm{RP}_h - x^c_{\mathbf{b}_h}(0)$ is the gain (utility) of $\mathbf{b}_h$ in complete information bargaining and $\mathrm{RP}_l - x^c_{\mathbf{b}_l}(0)$ is the gain (utility) of $\mathbf{b}_l$ in complete information bargaining.

Lemma 10. $x^c_{\mathbf{b}_h}(t) \leq (x^c_{\mathbf{b}_h}(t+1))_{\leftarrow \mathbf{b}_h}$ *and* $x^c_{\mathbf{b}_l}(t) \leq (x^c_{\mathbf{b}_l}(t+1))_{\leftarrow \mathbf{b}_l}$.

Proof. We can get this result by following the same procedure in the proof of Lemma 8. This result indicates that the buyer will accept sellers' lowest equilibrium price in complete information bargaining. □

Agents' optimal strategies when any seller $\mathbf{s}$ has the wrong belief about the type of the buyer $\mathbf{b}$ are shown in the following theorem:

Theorem 11. *If seller* $\mathbf{s}$ *has the wrong belief about the type of* $\mathbf{b}$*, the optimal strategies of any seller* $\mathbf{s}$ *are those in complete information bargaining. The optimal strategies* $\sigma^*_{\mathbf{b}_h}(t)|\{\mathbf{b}_l\}$ *of buyer* $\mathbf{b}_h$ *when it's believed to be* $\mathbf{b}_l$ *are:*

$$\sigma^*_{\mathbf{b}_h}(t)|\{\mathbf{b}_l\} = \begin{cases} \textit{accept } y & \text{if } y \leq (x^c_{\mathbf{b}_l}(t))_{\leftarrow \mathbf{b}_h} \\ \textit{offer } x^c_{\mathbf{b}_l}(t) & \textit{otherwise} \end{cases}$$

The optimal strategies $\sigma^*_{\mathbf{b}_l}(t)|\{\mathbf{b}_h\}$ *of the buyer* $\mathbf{b}_l$ *when it's believed to be* $\mathbf{b}_h$ *are:*

$$\sigma^*_{\mathbf{b}_l}(t)|\{\mathbf{b}_h\} = \begin{cases} \textit{accept } y & \text{if } y \leq \min\{(x^c_{\mathbf{b}_h}(t))_{\leftarrow \mathbf{b}_l}, \mathrm{RP}_l\} \\ \textit{offer } \min\{x^c_{\mathbf{b}_h}(t), \mathrm{RP}_l\} & \textit{otherwise} \end{cases}$$

Proof. Case 1 ($\mathbf{b}_h$ is believed to be $\mathbf{b}_l$). If sellers' lowest offer at time $t-2$ is $x^c_{\mathbf{b}_l}(t-1)$, buyer $\mathbf{b}_h$'s optimal strategy is to accept it as the minimum price that the seller would accept at time $t+1$, i.e., $x^c_{\mathbf{b}_l}(t)$, gives $\mathbf{b}_h$ a utility lesser than $x^c_{\mathbf{b}_l}(t-1)$ since $(x^c_{\mathbf{b}_l}(t))_{\leftarrow \mathbf{b}_h} > (x^c_{\mathbf{b}_l}(t))_{\leftarrow \mathbf{b}_l} \geq x^c_{\mathbf{b}_l}(t-1)$. If the seller acts off the equilibrium path and offers a price y lower than $x^c_{\mathbf{b}_l}(t-1)$, the optimal strategy of $\mathbf{b}_h$ is obviously to accept y. If the seller offers a price y greater than $x^c_{\mathbf{b}_l}(t-1)$, the optimal strategy of $\mathbf{b}_h$ is to accept y only if $y \leq (x^c_{\mathbf{b}_l}(t))_{\leftarrow \mathbf{b}_h}$, otherwise $\mathbf{b}_h$'s optimal strategy is to reject y and to offer $x^c_{\mathbf{b}_l}(t)$. Note that $x^c_{\mathbf{b}_h}(t) \leq \mathrm{RP}_h$ and $x^c_{\mathbf{b}_l}(t) \leq \mathrm{RP}_h$.

Case 2 ($\mathbf{b}_l$ is believed to be $\mathbf{b}_h$). This case is more complicated as sellers' lowest offer $x^c_{\mathbf{b}_h}(t-1)$ at time t on its equilibrium path may be not acceptable to $\mathbf{b}_l$ as when $\mathbf{b}_l$ offers $x^c_{\mathbf{b}_h}(t)$ at time t, it follows that $(x^c_{\mathbf{b}_h}(t))_{\leftarrow \mathbf{b}_l} < (x^c_{\mathbf{b}_h}(t))_{\leftarrow \mathbf{b}_h}$ and $(x^c_{\mathbf{b}_h}(t))_{\leftarrow \mathbf{b}_h} \geq x^c_{\mathbf{b}_h}(t-1)$ (Lemma 10). In addition, $\mathbf{b}_l$ may not offer $x^c_{\mathbf{b}_h}(t)$ if $x^c_{\mathbf{b}_h}(t)$ is higher than RP_l. Therefore, $\mathbf{b}_l$'s optimal offer at time t is $\min\{x^c_{\mathbf{b}_h}(t), \mathrm{RP}_l\}$. Thus, $\mathbf{b}_l$ will accept an offer y at time t such that $y \leq \min\{(x^c_{\mathbf{b}_h}(t))_{\leftarrow \mathbf{b}_l}, \mathrm{RP}_l\}$. □

5.3 Solving Algorithm

It is well known that backward induction cannot be employed with extensive-form games with incomplete information. Several works tried to extend it to this problem with pure strategies, but they were proved not to be sound. These works at first compute backward agents' optimal strategies assuming agents' beliefs to be the initial ones, and then, once strategies are computed, they compute belief system consistent with respect to the strategies. However, the computed strategies are not assured to be optimal anymore given the belief system.

Our algorithm combines game theoretical analysis and state space search techniques and it is sound and complete. By applying state space search, we enumerate all possible choice trees (systems). A choice tree specifies buyer types' choice rule class at all decision making points along the negotiation horizon. By exploiting game theoretical analysis we design a pair composed of choice rules and belief system for each possible class of choice rules. More precisely, we design a pair for pooling choice rule and a pair for separating choice rule. These pairs are parameterized: agents' optimal offers and acceptance at time t depend on the agents' strategies in the following time points till the end of the bargaining. Furthermore, we assign each pair some conditions: if they are satisfied, then there is a sequential equilibrium in the subgame starting from time t. For each choice tree, we employ a Bayesian extension of backward induction to derive agents' optimal strategies. Agents' optimal strategies at time t is built on agents' equilibrium strategies from time $t + 1$ to T. In summary, we employ a forward-backward approach to find sequential equilibria: we search forward to find all the choice trees (systems) and we construct backward agents' equilibrium strategies and belief systems.

Given $\mathbf{s}$'s belief on the type of $\mathbf{b}$, different buyer types can choose different choice rules: either behave in the same way or behave in different ways. While it is very involved to compute sequential equilibria considering all the options at each decision making point, we explicitly fix the choice rule at each decision making point and then compute the sequential equilibrium of the bilateral game where buyer types' choices are specified in the choice tree. To guarantee the completeness of our approach, we enumerate all possible choice trees. Our approach can be treated as a way of shifting the difficulty of finding a sequential equilibrium in a bargaining game where the buyer has multiple choices to finding a sequential equilibrium in multiple bargaining games in which the buyer's choice is fixed.

We explain our approach through a bilateral bargaining example with two-type uncertainty where $\iota(0) = \mathbf{b}$ and $T = 4$. As there are two types, there exist only two choice rules: 1) different buyer types behave in the same way (i.e., make the same offer) or 2) different buyer types behave in different ways (i.e., make different offers). Once the buyer chooses to differentiate its two types at time point t, the later bargaining becomes complete information bargaining. At time $t = 0$, the belief of $\mathbf{s}$ on the type of $\mathbf{b}$ is $\{\mathbf{b}_h, \mathbf{b}_l\}$ and $\mathbf{b}$'s different types can choose to make the same offer or make different offers. If $\mathbf{b}$ chooses the separating rule, seller $\mathbf{s}$ will update its belief at time $t = 1$ and bargaining from time $t = 1$ to the deadline becomes complete information bargaining. If $\mathbf{b}$ chooses the pooling rule at time $t = 0$, seller $\mathbf{s}$'s belief at time $t = 1$ will still be $\{\mathbf{b}_h, \mathbf{b}_l\}$. Then at time $t = 2$, buyer types can still choose to behave in the same way or behave in different ways. No matter what the choice rule is at time $t = 2$, $\mathbf{b}$ has no choice at its deadline $t = 4$. There are totally three choice trees: 1) different buyer types always use the pooling choice rule; 2) different buyer types apply the pooling choice rule at time $t = 0$ and apply the separating choice rule at time $t = 2$; and 3) different buyer types apply the separating choice rule at time $t = 0$.

Given a choice tree, we adopt a modified backward induction approach to compute the sequential equilibrium. In the rest of this section, we first consider two special situations where buyer types always choose the pooling choice rule or always choose the separating choice rule. Then we construct sequential equilibria for general cases based on our analysis of the two special situations.

5.4 Always Use Pooling Choice Rule

In this section we study agents' equilibrium strategies when different buyer types always behave in the same way at each decision making point. Accordingly, sellers' beliefs will always be its initial belief. We start considering a bargaining with deadline $T = 3$. There are two situations: $\iota(0) = \mathcal{S}$ or $\iota(0) = \mathbf{b}$. We first consider the former case in which both $\mathbf{b}_h$ and $\mathbf{b}_l$ will propose $x^*_{\mathbf{b}_h}(1) = x^*_{\mathbf{b}_l}(1) = \mathrm{RP}_{\mathcal{S}_3^1}$ at time $t = 1$. Any seller $\mathbf{s} \in \mathcal{S}_2$ will propose its best offer based on its initial belief at $t = 0$. It's easy to see that the in equilibrium 1) seller $\mathcal{S}_2^1$ will not propose a price higher than $\mathrm{RP}_{\mathcal{S}_2^2}$, and 2) sellers other than $\mathcal{S}_2^1$ will propose their reserve prices. If buyer $\mathbf{b}$ is of type $\mathbf{b}_h$, seller $\mathcal{S}_2^1$'s optimal offer is $x^*_{\mathbf{b}_h}(0) = \min\{\mathrm{RP}_{\mathcal{S}_2^2}, (x^*_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}\}$. If buyer $\mathbf{b}$ is of type $\mathbf{b}_l$, seller $\mathcal{S}_2^1$'s optimal offer is $x^*_{\mathbf{b}_l}(0) = \min\{\mathrm{RP}_{\mathcal{S}_2^2}, (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l}\}$. Thus, at time $t = 0$. $\mathcal{S}_2^1$ has two choices: 1) $x^*_{\mathbf{b}_l}(0)$ with an expected utility $EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_l}(0)) = U_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_l}(0), 2)$ since both buyer types will accept the offer $x^*_{\mathbf{b}_l}(0)$ at time $t = 1$; 2) $x^*_{\mathbf{b}_h}(0)$ with an expected utility $EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_h}(0))$ where $EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_h}(0)) = \omega^0_{\mathbf{b}_h} U_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_h}(0), 2) + \omega^0_{\mathbf{b}_l} U_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_l}(1), 3)$ if $\mathcal{S}_2^1$ has a deadline $T_{\mathcal{S}_2^1} = 3$ and $EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_h}(0)) = \omega^0_{\mathbf{b}_h} U_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_h}(0), 2) + 0$ if $\mathcal{S}_2^1$ has a deadline $T_{\mathcal{S}_2^1} = 2$.

With incomplete information, we need to introduce the notion of *equivalent* value (price) of an offer, which is the value to be propagated backward. In one-to-many negotiation, only the seller with the lowest reserve price can gain a positive utility. Thus, we only need to consider the equivalent price of the optimal offer of agent $\mathcal{S}_{t+2}^1$ at time t. Let $e^0_{\mathcal{S}_2^1}$ the equivalent price of the optimal offer of agent $\mathcal{S}_2^1$ in the subgame beginning from time 0 where it begins to bargain. Formally, $e^0_{\mathcal{S}_2^1}$ is a price such that $U_{\mathcal{S}_2^1}(e^0_{\mathcal{S}_2^1}, t+2) = \max\{EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_h}(0)), EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_l}(0))\}$. The negotiation outcome will be:

1. If $EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_l}(0)) \geq EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_h}(0))$, $\mathcal{S}_2^1$ will offer $x^*_{\mathbf{b}_l}(0)$ at time $t = 0$ and it will accepted by the buyer independent the its type.

2. If $EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_l}(0)) < EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_h}(0))$ and $\mathcal{S}_2^1$ has a deadline 3, $\mathcal{S}_2^1$ will offer $x^*_{\mathbf{b}_h}(0)$ at time $t = 0$ and it will accepted by the buyer if it is of type $\mathbf{b}_h$. Otherwise, buyer $\mathbf{b}_l$ will propose $x^*_{\mathbf{b}_l}(1)$ at time 1 and an agreement will be made at time 3 between buyer $\mathbf{b}_l$ and seller $\mathcal{S}_2^1$.

3. If $EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_l}(0)) < EU_{\mathcal{S}_2^1}(x^*_{\mathbf{b}_h}(0))$ and $\mathcal{S}_2^1$ has a deadline 2, $\mathcal{S}_2^1$ will offer $x^*_{\mathbf{b}_h}(0)$ at time $t = 0$ and it will accepted by the buyer if it is of type $\mathbf{b}_h$. Otherwise, buyer $\mathbf{b}_l$ will propose $x^*_{\mathbf{b}_l}(1)$ at time 1 and an agreement will be made at time 3 between buyer $\mathbf{b}_l$ and seller $\mathcal{S}_3^1$.

Now consider the case $\iota(0) = \mathbf{b}$. At time $t = 1$, sellers $\mathcal{S}_3$ will reason about their equilibrium (optimal) strategies. If $\mathcal{S}_3$ includes only one seller $\mathcal{S}_3^1$, it can choose between the offer RP_h and RP_l. While offering RP_h, it can get an expected utility $EU_{\mathcal{S}_3^1}(\mathrm{RP}_h) = \omega^0_{\mathbf{b}_h} U_{\mathcal{S}_3^1}(\mathrm{RP}_h, T)$ as $\mathbf{b}_l$ will not accept the offer RP_h. While offering RP_l, it can get an expected utility $EU_{\mathcal{S}_3^1}(\mathrm{RP}_l) = U_{\mathcal{S}_3^1}(\mathrm{RP}_l, T)$. Let $e^1_{\mathcal{S}_3^1}$ the equivalent price of the optimal offer of agent $\mathcal{S}_3^1$ in the subgame beginning from time 1 where it begins to bargain. Formally, $e^1_{\mathcal{S}_3^1}$ is a price such that $U_{\mathcal{S}_3^1}(e^1_{\mathcal{S}_3^1}, T) = \max\{EU_{\mathcal{S}_3^1}(\mathrm{RP}_h), EU_{\mathcal{S}_3^1}(\mathrm{RP}_l)\}$. If $\mathcal{S}_3$ includes more than one seller, seller $\mathcal{S}_3^1$'s optimal offer at time $t = 1$ is $\mathrm{RP}_{\mathcal{S}_3^2}$ due to the competition. Thus, the equivalent price of the optimal offer of agent $\mathcal{S}_3^1$ in this case is $\mathrm{RP}_{\mathcal{S}_3^2}$. Given $e^1_{\mathcal{S}_3^1}$, $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1}$ is the lowest offer agent $\mathcal{S}_3^1$ would accept at time $t = 1$ with the initial belief. Therefore, the optimal offer of the buyer at time $t = 0$ is

$$x^*_{\mathbf{b}}(0) = \begin{cases} \mathrm{RP}_{\mathcal{S}_2^1} & T_{\mathcal{S}_2^1} = 2 \\ \min\{(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1}, \mathrm{RP}_{\mathcal{S}_2^2}\} & T_{\mathcal{S}_2^1} \neq 2 \text{ and } |\mathcal{S}_2| > 1 \\ (e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1} & T_{\mathcal{S}_2^1} \neq 2 \text{ and } |\mathcal{S}_2| = 1 \end{cases}$$

Note that if $T_{\mathcal{S}_2^1} \neq 2$ (i.e., $\mathcal{S}_3^1 = \mathcal{S}_2^1$), $|\mathcal{S}_2| = 1$, and $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1} > \mathrm{RP}_l$, it is not rational for $\mathbf{b}_l$ to offer $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1}$. In this case, there is no sequential rational strategy while always using the pooling choice rule. If $|\mathcal{S}_2| > 1$, it is both buyer types to offer the above specified optimal price as it is impossible to have $\mathrm{RP}_{\mathcal{S}_2^2} > \mathrm{RP}_l$. Agents' equilibrium strategies when $T = 3$ and $\iota(T) = \mathcal{S}$ are specified in the following theorem.

Theorem 12. *Assume that $T = 3$ and $\iota(0) = \mathbf{b}$, if $\mathrm{RP}_l \geq (e^1_{\mathcal{S}_3^1})_{\leftarrow \mathbf{s}}$ when $T_{\mathcal{S}_2^1} \neq 2$ and $|\mathcal{S}_2| = 1$, there is one and only one sequentially rational pure strategy profile given the system of beliefs*

$$\mu(1) = \begin{cases} \Delta^0_{\mathbf{b}} & \text{if } \sigma^*_{\mathbf{b}}(0) = \textit{offer } x^*_{\mathbf{b}}(0) \\ \{\mathbf{b}_h\} & \textit{otherwise} \end{cases}$$

The strategies $\sigma^*_{\mathbf{b}_h}(0)$ and $\sigma^*_{\mathbf{b}_l}(0)$ are: $\sigma^*_{\mathbf{b}_h}(0) = \sigma^*_{\mathbf{b}_l}(0) = \textit{offer } x^*_{\mathbf{b}}(0)$. The strategy $\sigma^*_{\mathbf{s}}(1)$ is: 1) $\sigma^*_{\mathbf{s}}(1) = \textit{accept } y \text{ if } y \geq \mathrm{RP}_{\mathbf{s}}$ if $T_{\mathbf{s}} = 2$; 2) $\sigma^*_{\mathbf{s}}(1) = \textit{accept } y \text{ if } y \geq \max\{(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1}, \mathrm{RP}_{\mathbf{s}}\}$ if $T_{\mathbf{s}} \neq 2$, $T_{\mathcal{S}_2^1} \neq 2$, and $|\mathcal{S}_2| = 1$; $\sigma^*_{\mathbf{s}}(1) = \textit{accept } y \text{ if } y \geq \max\{\min\{(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1}, \mathrm{RP}_{\mathcal{S}_2^2}\}, \mathrm{RP}_{\mathbf{s}}\}$ if $T_{\mathbf{s}} \neq 2$, $T_{\mathcal{S}_2^1} \neq 2$, and $|\mathcal{S}_2| > 1$. $\mathbf{b}$ will confirm the agreement with the lowest reserve price at time $t = 2$.

Proof. We analyze the strategies on the equilibrium path. We assume that the buyer behaves according to the prescribed equilibrium strategies and we analyze the optimal strategy of the seller. There are three different situations. For any seller has a deadline 2, it will receive any offer no less than its reserve price. If there is only one seller which has a deadline 3, the seller can accept $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathbf{s}}$ and gain $U_{\mathbf{s}}((e^1_{\mathcal{S}_3^1})_{\leftarrow \mathbf{s}}, T-1)$ or reject it and make an offer. However, the maximum expected utility the seller can have from the subgame from time 1 is just $U_{\mathbf{s}}(e^1_{\mathcal{S}_3^1}, T) = U_{\mathbf{s}}((e^1_{\mathcal{S}_3^1})_{\leftarrow \mathbf{s}}, T-1)$. Thus, the seller's optimal strategy is to accept $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathbf{s}}$. In the third case (i.e., $T_{\mathbf{s}} \neq 2$, $T_{\mathcal{S}_2^1} \neq 2$, and $|\mathcal{S}_2| > 1$), seller $\mathcal{S}_3^1$ will accept the offer $\min\{(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1}, \mathrm{RP}_{\mathcal{S}_2^2}\}$ given that it cannot gain a higher utility by rejecting the offer $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1}$. In addition, given the competition between sellers, seller $\mathcal{S}_3^1$ has to accept the offer $\mathrm{RP}_{\mathcal{S}_2^2}$.

We assume that the sellers behave according to the prescribed equilibrium strategies and we analyze the optimal strategy of the buyer. There are three situations. If $T_{\mathcal{S}_2^1} = 2$, it's both buyer types' optimal strategy to offer $\mathrm{RP}_{\mathcal{S}_2^1}$. For the case $T_{\mathcal{S}_2^1} \neq 2$ and $|\mathcal{S}_2| = 1$, we start considering the strategy of $\mathbf{b}_h$. If $\mathbf{b}_h$ offers $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_2^1}$, it gains $U_{\mathbf{b}_h}((e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_2^1}, T-1)$. If $\mathbf{b}_h$ offers a price y higher than $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_2^1}$, the seller will reject it and propose the price $\mathrm{RP}_{\mathbf{b}_h}$. Then $\mathbf{b}_h$'s final utility $U_{\mathbf{b}_h}(\mathrm{RP}_{\mathbf{b}_h}, T)$ is not higher than $U_{\mathbf{b}_h}((e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_2^1}, T-1)$. If $\mathbf{b}_h$ proposes a price y lower than $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_2^1}$, the seller will accept it and it gains a utility lower than $U_{\mathbf{b}_h}((e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_2^1}, T-1)$. Similarly, we can get that $\mathbf{b}_l$ has no incentive to propose a price not equal to $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_2^1}$. In the same way, we can prove that the optimal offer for both buyer types is $\min\{(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_3^1}, \mathrm{RP}_{\mathcal{S}_2^2}\}$ in the situation such that $T_{\mathcal{S}_2^1} \neq 2$ and $|\mathcal{S}_2| > 1$.

□

If $\mathrm{RP}_l \geq (e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_2^1}$ when $T_{\mathcal{S}_2^1} \neq 2$ and $|\mathcal{S}_2| = 1$, there is no sequential rational pure strategy in our belief system since it supposes that both the buyer's types behave in the same way, whereas the optimal strategy of $\mathbf{b}_l$ is to not propose $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathcal{S}_2^1}$ as it will get a negative utility by doing so. Fig. 3 shows an example of backward induction construction with $T = 3$, $\iota(T) = \mathcal{S} = \{\mathbf{s}\}$, $\mathrm{RP}_h = 100$, $\omega^0_{\mathbf{b}_h} = 0.7$, $\mathrm{RP}_l = 60$, $\omega^0_{\mathbf{b}_l} = 0.3$, $\mathrm{RP}_{\mathbf{s}} = 10$, $\delta_{\mathbf{s}} = \delta_{\mathbf{b}} = 0.6$. At time $t = 1$, $\mathbf{s}$ can offer either 60 or 100: If it offers 60, its expected utility is $(60-10)0.6^2 = 18$; If it offers 100, its expected utility is $0.7(100-10)0.6^2 = 22.68$. Thus, the optimal offer of $\mathbf{s}$ at time $t = 1$ is 100 and the equivalent price is $e^1_{\mathcal{S}_3^1} = 73$. Then we have $(e^1_{\mathcal{S}_3^1})_{\leftarrow \mathbf{s}} = 47.8$. As $\mathrm{RP}_l > (e^1_{\mathcal{S}_3^1})_{\leftarrow \mathbf{s}}$, there is a sequential equilibrium within the belief system while always using the pooling choice rule. If we change RP_l to 30 (Fig. 4). At time $t = 1$, $\mathbf{s}$ can offer either 30 or 100. If it offers 30, its

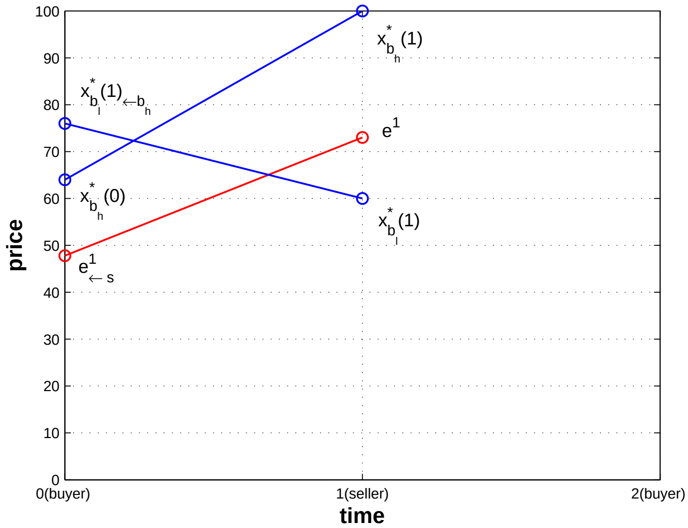

Figure 3: Backward induction construction with $T = 2$, $\iota(T) = \mathbf{b}$, $\mathrm{RP}_h = 100$, $\omega^0_{\mathbf{b}_h} = 0.7$, $\mathrm{RP}_l = 60$, $\omega^0_{\mathbf{b}_l} = 0.3$, $\mathrm{RP}_{\mathbf{s}} = 10$, $\delta_{\mathbf{s}} = \delta_{\mathbf{b}} = 0.6$.

expected utility is $(30 - 10)0.6^2 = 7.2$. Thus, the optimal offer of $\mathbf{s}$ at time $t = 1$ is 100 and the equivalent price is $e^1_{\mathcal{S}^1_3} = 73$. Then we have $(e^1_{\mathcal{S}^1_3})_{\leftarrow \mathbf{s}} = 47.8$. As $\mathrm{RP}_l < (e^1_{\mathcal{S}^1_3})_{\leftarrow \mathbf{s}}$, there is no sequential rational strategy within the belief system while always using the pooling choice rule.

We now consider an arbitrary deadline T. We apply the backward induction starting from deadline T and inductively determine agents' equilibrium strategies. Let $e^t_{\mathcal{S}^1_{t+2}}$ be the equivalent price of the optimal offer of $\mathcal{S}^1_{t+2}$ at time t when $\iota(t) = \mathcal{S}$. First we consider the case $\iota(T) = \mathcal{S}$. At any time t such that $\iota(t) = \mathbf{b}$, $\mathbf{b}$'s optimal offer is

$$x^*_{\mathbf{b}}(t) = \begin{cases} \mathrm{RP}_{\mathcal{S}^1_{t+2}} & T_{\mathcal{S}^1_{t+2}} = t+2 \\ \min\{(e^{t+1}_{\mathcal{S}^1_{t+3}})_{\leftarrow \mathcal{S}^1_{t+3}}, \mathrm{RP}_{\mathcal{S}^2_{t+2}}\} & T_{\mathcal{S}^1_{t+2}} \neq t+2 \text{ and } |\mathcal{S}_{t+2}| > 1 \\ (e^{t+1}_{\mathcal{S}^1_{t+3}})_{\leftarrow \mathcal{S}^1_{t+3}} & T_{\mathcal{S}^1_{t+2}} \neq t+2 \text{ and } |\mathcal{S}_{t+2}| = 1 \end{cases}$$

At time $T-2$, the equivalent price $e^{T-2}_{\mathcal{S}^1_T}$ is defined as follows. If $\mathcal{S}_T$ includes only one seller $\mathcal{S}^1_T$, it can choose between the offer RP_h and RP_l. While offering RP_h, it can get an expected utility $EU_{\mathcal{S}^1_T}(\mathrm{RP}_h) = \omega^0_{\mathbf{b}_h} U_{\mathcal{S}^1_T}(\mathrm{RP}_h, T)$ as $\mathbf{b}_l$ will not accept the offer RP_h. While offering RP_l, it can get an expected utility $EU_{\mathcal{S}^1_T}(\mathrm{RP}_l) = U_{\mathcal{S}^1_T}(\mathrm{RP}_l, T)$. $e^{T-2}_{\mathcal{S}^1_T}$ is a price such that $U_{\mathcal{S}^1_{T-2}}(e^{T-2}_{\mathcal{S}^1_T}, T) = \max\{EU_{\mathcal{S}^1_{T-2}}(\mathrm{RP}_h), EU_{\mathcal{S}^1_{T-2}}(\mathrm{RP}_l)\}$. If $\mathcal{S}_T$ includes more than one seller, seller $\mathcal{S}^1_T$'s optimal offer at time $T-2$ is $\mathrm{RP}_{\mathcal{S}^2_T}$ due to the competition. Thus, the equivalent price of the optimal offer of agent $\mathcal{S}^1_T$ in this case is $e^{T-2}_{\mathcal{S}^1_T} = \mathrm{RP}_{\mathcal{S}^2_T}$.

At time $t < T-2$, there are two cases. If there is only one seller in $\mathcal{S}_{t+2}$, i.e., $\mathcal{S}_{t+2} = \{\mathcal{S}^1_{t+2}\}$, $\mathcal{S}^1_{t+2}$ has two choices: propose $(x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}$ or $(x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}$. In this case, $e^t_{\mathcal{S}^1_{t+2}}$ satisfies $U_{\mathcal{S}^1_{t+2}}(e^t_{\mathcal{S}^1_{t+2}}, t+2) = \max\{EU_{\mathcal{S}^1_{t+2}}((x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}), EU_{\mathcal{S}^1_{t+2}}((x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l})\}$ where $EU_{\mathcal{S}^1_{t+2}}((x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}) = U_{\mathcal{S}^1_{t+2}}((x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}, t+2)$ and $EU_{\mathcal{S}^1_{t+2}}((x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}) = \omega^0_{\mathbf{b}_h} U_{\mathcal{S}^1_{t+2}}((x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}, t+2) + \omega^0_{\mathbf{b}_l} U_{\mathcal{S}^1_{t+2}}(x^*_{\mathbf{b}}(t+1), t+3)$.

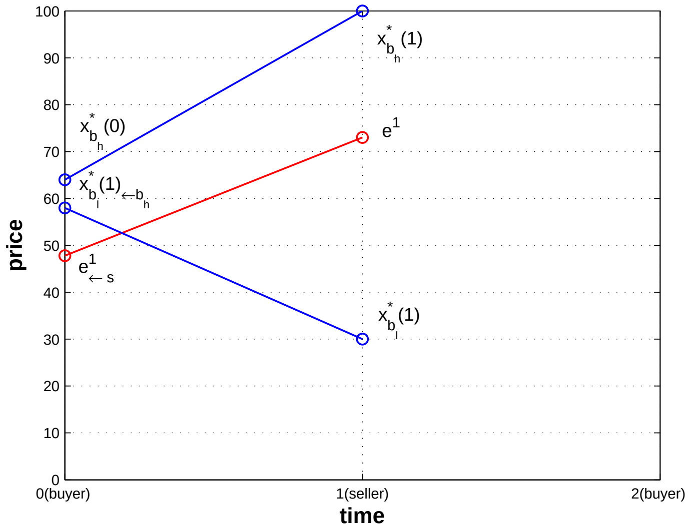

Figure 4: Backward induction construction with the same setting as in Fig. 3 except $\mathrm{RP}_l = 30$.

If there is more than one seller in $\mathcal{S}_{t+2}$, i.e., $|\mathcal{S}_{t+2}| > 1$, $\mathcal{S}^1_{t+2}$ has two choices: propose $\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}\}$ or $\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}\}$. In this case, $e^t_{\mathcal{S}^1_{t+2}}$ satisfies $U_{\mathcal{S}^1_{t+2}}(e^t_{\mathcal{S}^1_{t+2}}, t+2) = \max\{EU_{\mathcal{S}^1_{t+2}}(\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}\}), EU_{\mathcal{S}^1_{t+2}}(\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}\})\}$ where $EU_{\mathcal{S}^1_{t+2}}(\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}\}) = U_{\mathcal{S}^1_{t+2}}(\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}\}, t+2)$ and $EU_{\mathcal{S}^1_{t+2}}(\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, \min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}\}\}) = \omega^0_{\mathbf{b}_h} U_{\mathcal{S}^1_{t+2}}(\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, \min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}\}\}, t+2) + \omega^0_{\mathbf{b}_l} U_{\mathcal{S}^1_{t+2}}(x^*_{\mathbf{b}}(t+1), t+3)$.

Now we consider the case $\iota(T) = \mathbf{b}$. $\mathbf{b}$'s optimal offer $x^*_{\mathbf{b}}(T-2)$ at time $T-2$ is $\mathrm{RP}_{\mathcal{S}^1_T}$. $\mathbf{b}$'s optimal offer $x^*_{\mathbf{b}}(t)$ and equivalent price $e^t_{\mathcal{S}^1_{t+2}}$ at time $t < T-2$ can be calculated in the same way as in the case $\iota(T) = \mathcal{S}$. Following Theorem 12, the condition of existence includes $\mathrm{RP}_l \geq x^*_{\mathbf{b}}(t)$ and $x^*_{\mathbf{b}}(t) \leq (x^*_{\mathbf{b}}(t'))_{\leftarrow (t'-t)[\mathbf{b}_l]}$ at any time $t < T-1$, i.e., the optimal offer at time t is better than the later optimal offers.

Theorem 13. *The one-to-many bargaining has a unique sequentially rational pure strategy profile given the following belief system where $\mu(t+1)$ is given by*

- *If $\mu(t) = \{\mathbf{b}_h\}$ or $\mu(t) = \{\mathbf{b}_l\}$, $\mu(t+1) = \mu(t)$.*
- *$\mu(t) = \mu(0)$ and there are four cases. 1) If $t = 0$ and $\sigma_{\mathbf{b}}(t) =$ offer$x^*_{\mathbf{b}}(t)$, $\mu(t+1) = \mu(0) = \{\mathbf{b}_h, \mathbf{b}_l\}$; 2) If $t > 0$ and $\mathbf{b}$ rejects $y \in ((x^*_{\mathbf{b}}(t))_{\leftarrow \mathbf{b}_h}, +\infty]$ and $\sigma_{\mathbf{b}}(t) =$ offer $x^*_{\mathbf{b}}(t)$, $\mu(t+1) = \mu(0) = \{\mathbf{b}_h, \mathbf{b}_l\}$; 3) If $t > 0$ and $\mathbf{b}$ rejects $y \in ((x^*_{\mathbf{b}}(t))_{\leftarrow \mathbf{b}_l}, (x^*_{\mathbf{b}}(t))_{\leftarrow \mathbf{b}_h}]$ and $\sigma_{\mathbf{b}}(t) =$ offer $x^*_{\mathbf{b}}(t)$, $\mu(t+1) = \{\mathbf{b}_l\}$; 4) otherwise, $\mu(t+1) = \{\mathbf{b}_h\}$.*

if $\mathrm{RP}_l \geq x^*_{\mathbf{b}}(t)$ and $x^*_{\mathbf{b}}(t) \leq (x^*_{\mathbf{b}}(t'))_{\leftarrow (t'-t)[\mathbf{b}_l]}$ at any time $t < T-1$. The equilibrium strategies $\sigma^*_{\mathbf{s}}(t)|\{\mathbf{b}_h, \mathbf{b}_l\}$ of agent $\mathbf{s}$ are:

$$\begin{cases} \textit{accept } y & \text{if } y \geq x^*_{\mathbf{b}}(t) \\ \textit{offer } \min\{\mathrm{RP}_{\mathbf{s}}, \arg\max_{y \in \{(x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}\}} EU_{\mathbf{s}}(y) & \text{if } y < x^*_{\mathbf{b}}(t) \text{ and } |\mathcal{S}_{t+2}| = 1 \\ \textit{offer } \min\{\mathrm{RP}_{\mathbf{s}}, \arg\max_{y \in \{\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}\}, \min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}\}\}}\} EU_{\mathbf{s}}(y) & \text{if } y < x^*_{\mathbf{b}}(t) \text{ and } |\mathcal{S}_{t+2}| > 1 \end{cases}$$

and the equilibrium strategies of the buyer are:

$$\sigma^*_{\mathbf{b}_h}(t)|\{\mathbf{b}_h,\mathbf{b}_l\} = \begin{cases} accept\ y & \text{if } y \le (x^*_{\mathbf{b}}(t))_{\leftarrow \mathbf{b}_h} \\ offer\ x^*_{\mathbf{b}}(t) & otherwise \end{cases}$$

$$\sigma^*_{\mathbf{b}_l}(t)|\{\mathbf{b}_h,\mathbf{b}_l\} = \begin{cases} accept\ y & \text{if } y \le (x^*_{\mathbf{b}}(t))_{\leftarrow \mathbf{b}_l} \\ offer\ x^*_{\mathbf{b}}(t) & otherwise \end{cases}$$

Agents' equilibrium strategies when $\mu(t)$ is a singleton is given by Section 5.2.

Proof. The sequential rationality is easily seen from the backward construction. Consistency can be proved by the assessment sequence $a_n = (\mu_n, \sigma_n)$ where σ_n is the fully mixed strategy profile such that for the sellers and $\mathbf{b}_h$ there is probability $1 - 1/n$ of performing the action prescribed by the equilibrium strategy profile and the remaining probability $1/n$ is uniformly distributed among the other allowed actions; while for $\mathbf{b}_l$, there is probability $1 - 1/n^2$ of performing the action prescribed by the equilibrium strategy profile and the remaining probability $1/n^2$ is uniformly distributed among the other allowed actions, and μ_n is the system of beliefs obtained applying Bayes rule starting from the same *priori* probability distribution $P^0_{\mathbf{b}}$. As $n \to \infty$, the above mixed strategy profile converges to the equilibrium strategy profile. In addition, the beliefs generated by the mixed strategy profile converges to the priori probability distribution. Thus, the assessment is consistent. □

5.5 Always Use Separating Choice Rule

In this section we consider a different belief system in which two buyer types behave in different ways. Then the sellers will learn the buyer's type after it observes the buyer's first offer. Therefore, if $\iota(0) = \mathbf{b}$ ($\iota(0) = \mathcal{S}$, respectively), each seller $\mathbf{s}$ will learn $\mathbf{b}$'s type at beginning of time point $t = 1$ ($t = 2$, respectively) and the later bargaining is complete information bargaining.

We start by considering a bargaining with an arbitrary deadline T and $\iota(0) = \mathbf{b}$. Different from the approach in the previous section where we start backward induction from the deadline, we move forward from time $t = 0$. Let the equilibrium offers of $\mathbf{b}_h$ and $\mathbf{b}_l$ at time 0 be x and y such that $x \neq y$. If seller $\mathcal{S}^1_2$ accepts both offers x and y, at least one type has an incentive to offer $\min\{x, y\}$. Therefore, the offer $\min\{x, y\}$ will be rejected by $\mathcal{S}^1_2$. There are two cases: $x > y$ and $x < y$. First we consider the case $x < y$. Then $\mathbf{b}_h$ will make a low offer (e.g., $\mathrm{RP}_{\mathcal{S}^1_2}$) which be rejected by $\mathcal{S}^1_2$. Then at time 1, $\mathcal{S}^1_3$ will make the offer $x^c_{\mathbf{b}_h}(1)$ and $\mathbf{b}_h$ will accept it. The optimal offer $x^*_{\mathbf{b}_l}(0)$ of $\mathbf{b}_l$ at time $t = 0$ is

$$x^*_{\mathbf{b}_l}(0) = \begin{cases} \mathrm{RP}_{\mathcal{S}^1_2} & T_{\mathcal{S}^1_2} = 2 \\ \min\{(x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathcal{S}^1_2}, \mathrm{RP}_{\mathcal{S}^2_2}\} & T_{\mathcal{S}^1_2} \neq 2 \text{ and } |\mathcal{S}_2| > 1 \\ (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathcal{S}^1_2} & T_{\mathcal{S}^1_2} \neq 2 \text{ and } |\mathcal{S}_2| = 1 \end{cases}$$

We can find that $x^*_{\mathbf{b}_l}(0) = x^c_{\mathbf{b}_l}(0)$. $\mathbf{b}_h$ has no incentive to behave as $\mathbf{b}_l$ if $(x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h} < x^c_{\mathbf{b}_l}(0)$. There are two situations. If $T_{\mathcal{S}^1_2} = 2$, $x^c_{\mathbf{b}_l}(0) = \mathrm{RP}_{\mathcal{S}^1_2} < \mathrm{RP}_{\mathcal{S}^1_3} \le (\mathrm{RP}_{\mathcal{S}^1_3})_{\leftarrow \mathbf{b}_h} \le (x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}$. If $T_{\mathcal{S}^1_2} \neq 2$, we have $x^c_{\mathbf{b}_l}(0) \le (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathcal{S}^1_2} \le (x^c_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l} \le (x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}$. Thus, if $x < y$, the equilibrium is not sequential rational as $\mathbf{b}_h$ has an incentive to behave as $\mathbf{b}_l$.

Then we consider the case $x > y$. By convention that the equilibrium offer of $\mathbf{b}_l$ is $\mathrm{RP}_{\mathcal{S}^1_2}$ which will be rejected by each seller. The optimal offer of $\mathbf{b}_h$ is the lowest price agent $\mathcal{S}^1_2$ would accept at time 1 believing its opponent $\mathbf{b}_h$ is obviously $x^c_{\mathbf{b}_h}(0)$. The existence of a such equilibrium depends on two conditions: $\mathbf{b}_h$ must have no incentive to behave as $\mathbf{b}_l$, i.e., $x^c_{\mathbf{b}_h}(0) \le (x^c_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$, and $\mathbf{b}_l$ must have no incentive to behave as $\mathbf{b}_h$, i.e., $(x^c_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l} \le x^c_{\mathbf{b}_h}(0)$.

Theorem 14. *One-to-many bargaining such that $\iota(0) = \mathbf{b}$ has one and only one stationary sequential equilibrium profile in pure strategies given the system of beliefs:*

$$\mu(1) = \begin{cases} \{\mathbf{b}_l\} & \text{if } \sigma_{\mathbf{b}}(0) = \textit{offer}\, \mathrm{RP}_{\mathcal{S}_2^1} \\ \{\mathbf{b}_h\} & \textit{otherwise} \end{cases}$$

if $x^c_{\mathbf{b}_h}(0) \leq (x^c_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$ and $(x^c_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l} \leq x^c_{\mathbf{b}_h}(0)$. The equilibrium strategies of agent $\mathbf{b}$ are: $\sigma^*_{\mathbf{b}_h}(0)|\{\mathbf{b}_h, \mathbf{b}_l\} =$ offer $x^c_{\mathbf{b}_h}(0)$, $\sigma^*_{\mathbf{b}_l}(0)|\{\mathbf{b}_h, \mathbf{b}_l\} =$ offer $\mathrm{RP}_{\mathcal{S}_2^1}$. Agents' strategies when $\mu(t)$ is singleton are specified in Section 5.2.

Now we consider the case $\iota(0) = \mathcal{S}$. Sellers $\mathcal{S}_2$ know that at time $t = 1$ $\mathbf{b}_h$ will offer $x^c_{\mathbf{b}_h}(1)$ and $\mathbf{b}_l$ will offer offer $\mathrm{RP}_{\mathcal{S}_3^1}$. If there is only one seller in $\mathcal{S}_2$, $\mathcal{S}_2^1$ has two choices: propose $(x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}$ or $(x^c_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}$. In this case, $e^0_{\mathcal{S}_2^1}$ satisfies $U_{\mathcal{S}_2^1}(e^0_{\mathcal{S}_2^1}, 2) = \max\{EU_{\mathcal{S}_2^1}((x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}), EU_{\mathcal{S}_2^1}((x^c_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]})\}$ where $EU_{\mathcal{S}_2^1}(x^c_{\mathbf{b}_h}(1)_{\leftarrow \mathbf{b}_h}) = \omega^0_{\mathbf{b}_h} U_{\mathcal{S}_2^1}((x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}, 2) + \omega^0_{\mathbf{b}_l} U_{\mathcal{S}_2^1}(x^c_{\mathbf{b}_l}(2), 4)$ and $EU_{\mathcal{S}_2^1}((x^c_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}) = U_{\mathcal{S}_2^1}((x^c_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}, 2)$. If there is more than one seller in $\mathcal{S}_2$, i.e., $|\mathcal{S}_{t+2}| > 1$, $\mathcal{S}_2^1$ has two choices: propose $\min\{\mathrm{RP}_{\mathcal{S}_2^2}, (x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}\}$ or $\min\{\mathrm{RP}_{\mathcal{S}_2^2}, (x^c_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}\}$. In this case, $e^0_{\mathcal{S}_2^1}$ satisfies $U_{\mathcal{S}_2^1}(e^0_{\mathcal{S}_2^1}, 2) = \max\{EU_{\mathcal{S}_2^1}(\min\{\mathrm{RP}_{\mathcal{S}_2^2}, (x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}\}), EU_{\mathcal{S}_2^1}((x^c_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]})\}$ where $EU_{\mathcal{S}_2^1}(\min\{\mathrm{RP}_{\mathcal{S}_2^2}, (x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}\}) = \omega^0_{\mathbf{b}_h} U_{\mathcal{S}_2^1}(\min\{\mathrm{RP}_{\mathcal{S}_2^2}, (x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h}\}, 2) + \omega^0_{\mathbf{b}_l} U_{\mathcal{S}_2^1}(x^c_{\mathbf{b}_l}(2), 4)$ and $EU_{\mathcal{S}_2^1}(\min\{\mathrm{RP}_{\mathcal{S}_2^2}, (x^c_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}\}) = U_{\mathcal{S}_2^1}(\min\{\mathrm{RP}_{\mathcal{S}_2^2}, (x^c_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}\}, 2)$. $\mathcal{S}_2^1$ will choose the offer which gives it the highest expected utility at time 0.

Theorem 15. *Assume that the following belief system is used: if $\mathbf{b}$ rejects sellers' offer $y \in ((x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}, +\infty)$ and offers $\mathrm{RP}_{\mathcal{S}_2^1}$ at time 1, then $\mu(2) = \{\mathbf{b}_l\}$. Otherwise, $\mu(2) = \{\mathbf{b}_h\}$. One-to-many bargaining such that $\iota(0) = \mathcal{S}$ has a unique stationary sequential equilibrium profile in pure strategies if $x^c_{\mathbf{b}_h}(1) \leq (x^c_{\mathbf{b}_l}(2))_{\leftarrow \mathbf{b}_h}$ and $(x^c_{\mathbf{b}_l}(2))_{\leftarrow \mathbf{b}_l} \leq x^c_{\mathbf{b}_h}(1)$. The equilibrium strategies of agents are:*

$$\sigma^*_{\mathbf{s}}(0)|\{\mathbf{b}_h, \mathbf{b}_l\} = \textit{offer} \min\{\mathrm{RP}_{\mathbf{s}}, \arg \max_{y \in \{\min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h}\}, \min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_l}\}\}} EU_{\mathbf{s}}(y)\}$$

$$\sigma^*_{\mathbf{b}_h}(1)|\{\mathbf{b}_h, \mathbf{b}_l\} = \begin{cases} \textit{accept } y & \text{if } y \leq (x^c_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h} \\ \textit{offer } x^*_{\mathbf{b}_h}(1) & \textit{otherwise} \end{cases}$$

$$\sigma^*_{\mathbf{b}_l}(1)|\{\mathbf{b}_h, \mathbf{b}_l\} = \begin{cases} \textit{accept } y & \text{if } y \leq (x^c_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]} \\ \textit{offer } \mathrm{RP}_{\mathcal{S}_2^1} & \textit{otherwise} \end{cases}$$

Agents' strategies when $\mu(t)$ is singleton are those in complete information bargaining.

We can observe that the conditions for the existence of the above equilibrium are defined at the beginning of bargaining and the existence of the above equilibrium does not require the existence of a such equilibrium in later negotiation. Consider the sequential equilibrium when buyer types always choose different actions in the bilateral bargaining in Fig. 3. We have $x^c_{\mathbf{b}_h}(0) = 64$ and $x^c_{\mathbf{b}_l}(1) = 60$. $\mathbf{b}_h$ has no incentive to behave as $\mathbf{b}_l$ since $x^c_{\mathbf{b}_h}(0) = 64 < 76 = (x^c_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$, and $\mathbf{b}_l$ has no incentive to behave as $\mathbf{b}_h$ since $(x^c_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l} = 60 < 64 = x^c_{\mathbf{b}_h}(0)$. However, there is no sequential equilibrium within the belief system for the bilateral bargaining in Fig. 4. We have $x^c_{\mathbf{b}_h}(0) = 64$ and $x^c_{\mathbf{b}_l}(1) = 30$. However, this strategy is not rational for $\mathbf{b}_h$ as it has an incentive to behave as prescribed for $\mathbf{b}_l$ since $x^c_{\mathbf{b}_h}(0) = 64 \geq 58 = (x^c_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$.

5.6 Combining Pooling and Separating Choice Rules

In this section we consider agents' equilibrium strategies while employing a belief system which combines the pooling choice rule and the separating choice rule. The only reasonable combination is to employ the pooling choice rule from time 0 to some time $\tau \leq T$ and to employ the separating choice rule from time τ to the deadline T. The reason is simple: once different buyer types behave in different ways, each seller $\mathbf{s}$ will learn the type of the buyer and then agents conduct complete information bargaining. Then we have the following result and its proof is trivial.

Theorem 16. *For a finite horizon bargaining with two possible reserve prices of the buyer, if there is no sequential equilibrium in pure strategies, there is no sequential equilibrium in pure strategies within the belief system which employs both pooling and separating choice rules.*

Theorem 17. *For a finite horizon bargaining with two possible reserve prices of the buyer, there may be no sequential equilibrium in pure strategies.*

Proof. As the deadline of the bilateral bargaining in Fig. 4 is 2, if there is a sequential equilibrium in pure strategies, the choice rule at time $t = 0$ can only be pooling or separating. As there is no pure strategy sequential equilibrium in both cases, there is no pure strategy sequential equilibrium while applying both choice rules for the bilateral bargaining in Fig. 4. Thus, there is no pure strategy sequential equilibrium for the bilateral bargaining in Fig. 4 (Theorem 16). □

There may be more than one sequential equilibrium for a bilateral bargaining problem with two possible types of reserve price. For example, there are two sequential equilibria for the bilateral bargaining in Fig. 3: one with only using the pooling choice rule and one with only using the separating choice rule.

If there is a pure strategy sequential equilibrium, there should be a time point τ such that there is a sequential equilibrium for subgame $\Gamma^{[\tau,T]}$ which only uses the separating choice rule and a sequential equilibrium for subgame $\Gamma^{[0,\tau]}$ which only uses the pooling choice rule. Let the system of beliefs and equilibrium strategies for the subgame $\Gamma^{[\tau,T]}$ be $\mu^{[\tau,T]}$ and $\sigma^{*,[\tau,T]}$, respectively. Let the system of beliefs and equilibrium strategies for the subgame $\Gamma^{[0,\tau]}$ be $\mu^{[0,\tau]}$ and $\sigma^{*,[0,\tau]}$, respectively. The two equilibria form a sequential equilibrium.

Theorem 18. *If there is a τ such that*

1. *$\iota(\tau) = \mathcal{S}$;*
2. *There is a separating choice rule based sequential equilibrium $\langle \mu^{[\tau,T]}, \sigma^{*,[\tau,T]} \rangle$ for subgame $\Gamma^{[\tau,T]}$. Let e^{τ} be $\mathcal{S}^1_{t+2}$'s equivalent price at time τ;*
3. *There is a pooling choice rule based sequential equilibrium $\langle \mu^{[0,\tau]}, \sigma^{*,[0,\tau]} \rangle$ for subgame $\Gamma^{[0,\tau]}$ such that $\mathcal{S}^1_{\tau+2}$ accepts $x^*_{\mathbf{b}}(\tau-1)$ at time τ;*

$$x^*_{\mathbf{b}}(\tau-1) = \begin{cases} \mathrm{RP}_{\mathcal{S}^1_{\tau+1}} & T_{\mathcal{S}^1_{\tau-1}} = \tau+1 \\ \min\{(e^{\tau}_{\mathcal{S}^1_{\tau+2}})_{\leftarrow \mathcal{S}^1_{\tau+1}}, \mathrm{RP}_{\mathcal{S}^2_{\tau+1}}\} & T_{\mathcal{S}^1_{\tau+1}} \neq \tau+1 \text{ and } |\mathcal{S}_{\tau+1}| > 1 \\ (e^{\tau}_{\mathcal{S}^1_{\tau+2}})_{\leftarrow \mathcal{S}^1_{\tau+1}} & T_{\mathcal{S}^1_{\tau+1}} \neq \tau+1 \text{ and } |\mathcal{S}_{\tau+1}| = 1 \end{cases}$$

then $\langle \{\mu^{[0,\tau]}, \mu^{[\tau,T]}\}, \{\sigma^{*,[0,\tau]}, \sigma^{*,[\tau,T]}\} \rangle$ form a pure strategy sequential equilibrium.

The proof is omitted: Sequential rationality is obvious given the backward induction construction and consistency can be proved in the same way as in Theorem 13.

If there is a sequential equilibrium for such a τ value in Theorem 18, the equilibrium is unique for the specific τ given the backward induction process. Therefore, to find out a sequential equilibrium, we just need to search all the possible values of $\tau \leq T$. If there is no sequential equilibrium for all values of τ, we can conclude that there is no sequential equilibrium. The computational complexity of finding a sequential equilibrium for a specific value of τ is $\mathcal{O}(T)$. Thus, the computational complexity of finding sequential equilibrium for a bilateral bargaining with two possible types of reserve price is $\mathcal{O}(T^2)$. We can observe that the complexity of finding sequential equilibrium is independent of the number of sellers $|\mathcal{S}|$.

We show how to compute agents' equilibrium offers on the equilibrium path while using both the pooling choice rule and separating choice rule. We use the example in Fig. 3 and change the deadline to $T = 5$. First we consider the choice tree in which the pooling choice rule is used at both time $t = 0$ and $t = 2$. The optimal offer of $\mathbf{s}$ at time $t = 3$ is 100 and the equivalent price is $e^3 = 73$. At $t = 2$, both buyer type will offer $(e^3)_{\leftarrow \mathbf{s}} = 47.8$. At $t = 1$, $\mathbf{s}$ can offer 1) $(47.8)_{\leftarrow \mathbf{b}_h} = 68.68$, which will give $\mathbf{s}$ an expected utility with $0.7(68.68 - 10)0.6^2 + 0.3(47.8 - 10)0.6^3 = 17.2368$; 2) $(47.8)_{\leftarrow \mathbf{b}_l} = 52.68$, which will give $\mathbf{s}$ an expected utility with $(52.68 - 10)0.6^2 = 15.3648$. Therefore, the optimal offer of $\mathbf{s}$ at $t = 1$ is $(47.8)_{\leftarrow \mathbf{b}_h} = 68.68$ and the equivalent price is $e^1_{\mathcal{S}^1_3} = 57.88$. At $t = 0$, both buyer type will offer $(e^1_{\mathcal{S}^1_3})_{\leftarrow \mathbf{s}} = 38.728$. It's easy to see that all equilibrium existence conditions are satisfied. Thus, there is a sequential equilibrium with the choice tree.

Next we consider the choice tree in which the separating choice rule is used at time $t = 0$. First we assume the existence of sequential equilibrium and we have $x^*_{\mathbf{b}_h}(0) = 51.04$ and $x^*_{\mathbf{b}_l}(1) = 48$. $\mathbf{b}_h$ has no incentive to behave as $\mathbf{b}_l$ since $x^*_{\mathbf{b}_h}(0) = 51.04 < 68.8 = (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$. However, $\mathbf{b}_l$ has an incentive to behave as $\mathbf{b}_h$ since $(x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l} = 52.8 > 51.04 = x^*_{\mathbf{b}_h}(0)$. Therefore, there is no sequential equilibrium with this choice tree.

Finally, we consider the choice tree in which the pooling choice rule is used at time $t = 0$ and the separating choice rule is used at time $t = 2$. We first consider the subgame starting from $t = 2$, which is equivalent to the bargaining game in Fig. 3. Thus, there is a sequential equilibrium for the subgame with the separting choice rule in which $\mathbf{b}_h$'s optimal offer at time $t = 2$ is $x^*_{\mathbf{b}_h}(2) = 64$, $\mathbf{b}_l$'s optimal offer at time $t = 2$ is $\mathrm{RP}_{\mathbf{s}}$, and $\mathbf{s}$ will offer $x^*_{\mathbf{b}_l}(3) = 60$ at time $t = 3$ if it receives offer $\mathrm{RP}_{\mathbf{s}}$ at time $t = 2$. Then we consider the subgame from the beginning to time $t = 2$. At $t = 1$, $\mathbf{s}$ can offer 1) $(64)_{\leftarrow \mathbf{b}_h} = 78.4$, which will give $\mathbf{s}$ an expected utility $0.7(78.4 - 10)0.6^2 + 0.3(60 - 10)0.6^4 = 19.1808$ (note that if $\mathbf{b}$ is of type $\mathbf{b}_l$, it will offer $\mathrm{RP}_{\mathbf{s}}$ at time $t = 2$ and make an agreement with $\mathbf{s}$ at time $t = 4$); 2) $(x^*_{\mathbf{b}_l}(3))_{\leftarrow 2[\mathbf{b}_l]} = 60$, which will give $\mathbf{s}$ an expected utility $(60 - 10)0.6^2 = 18$. Therefore, the optimal offer of $\mathbf{s}$ at $t = 1$ is 78.4 and the equivalent price is $e^1_{\mathcal{S}^1_3} = 63.28$. At $t = 0$, both buyer type will offer $(e^1_{\mathcal{S}^1_3})_{\leftarrow \mathbf{s}} = 41.968$, which is lower than both types' reserve prices. Thus, there is a sequential equilibrium within this choice tree.

5.7 Multiple Types of Reserve Prices

In this section we extend our analysis for two-type uncertainty to finitely many type reserve prices for the buyer $\mathbf{b}$. The presence of many types increases the computational complexity of the procedure to find equilibrium strategies and requires more stringent equilibrium existence conditions. When there are only two types and the two buyer types behave in different ways at a time point, the only possibility is that the type with higher reserve price offers an acceptable price and the other type offer a price that will be rejected. With more types, the buyer has more options of differentiating its types: some buyer types make an acceptable offer while the other buyer types make an offer that will be rejected by the sellers. The number of the type partitions will exponentially increase with the increase of buyer types.

We show the complication introduced by the presence of multiple types through the following example: The initial belief of each seller $\mathbf{s}$ on the type of $\mathbf{b}$ is $\mu(0) = \langle \Delta^0_{\mathbf{b}}, P^0_{\mathbf{b}} \rangle$ where $\Delta^0_{\mathbf{b}} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. We assume that the deadline of each agent is 5 and $\iota(0) = \mathbf{b}$. At time $t = 0$, buyer $\mathbf{b}$ has the following options (choice rules): 1) $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ make the same offer; 2) One buyer type makes an offer acceptable to seller $\mathcal{S}^1_2$, while the other two buyer types make an offer that will be rejected by $\mathcal{S}^1_2$; and 3) Two buyer types make an offer acceptable to seller $\mathcal{S}^1_2$, while the other buyer type makes an offer that will be rejected by $\mathcal{S}^1_2$. At time $t = 1$, sellers will act according to its updated belief about the type of $\mathbf{b}$. For example, if $\mathbf{b}$ chooses option 1, each seller $\mathbf{s}$'s belief at time $t = 1$ is $\Delta^0_{\mathbf{b}}$. If $\mathbf{b}$ chooses one choice rule belonging to option 2 in which $\mathbf{b}_1$ and $\mathbf{b}_2$ make an offer acceptable to seller $\mathcal{S}^1_2$, but $\mathbf{b}_3$ makes an offer that will be rejected by $\mathcal{S}^1_2$, a seller $\mathbf{s}$'s belief at time $t = 1$ is either $\{\mathbf{b}_3\}$ or $\{\mathbf{b}_1, \mathbf{b}_2\}$.

If buyer $\mathbf{b}$ chooses option 1 at time $t = 0$, it still has three options at time $t = 2$ as at time $t = 0$. Assume that buyer $\mathbf{b}$ chooses the choice rule in which $\mathbf{b}_1$ and $\mathbf{b}_2$ make an offer acceptable to seller $\mathcal{S}^1_2$, but $\mathbf{b}_3$ makes an offer that will be rejected by $\mathcal{S}^1_2$. For the sellers' belief set (information set) $\{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathbf{b}$ has two options: 1) $\mathbf{b}_1$ and $\mathbf{b}_2$ make the same offer; 2) one buyer type makes an offer acceptable to seller $\mathcal{S}^1_2$, while the other makes an offer that will be rejected by $\mathcal{S}^1_2$. When a belief set contains only one type, bargaining becomes complete information bargaining.

Therefore, we can compute agents' sequential equilibrium strategies for the multi-type case based on our analysis of the two-type situation. Our approach involves searching all possible choice trees (each specifying agents' choice rule at each time point when the buyer is offering, i.e., whether different buyer types will behave in the same way or not) and computing agents' optimal strategies for each choice tree.

5.8 Many-to-many Setting

In this section we consider extending our analysis of one-to-many incomplete information bargaining to many-to-many incomplete information bargaining in which there is a two-type uncertainty regarding the type of a buyer $\mathbf{b} \in \mathcal{B}$ while the types of other buyers $\mathcal{B} - \mathbf{b}$ and sellers $\mathcal{S}$ are known to all agents. Adding other buyers will change the strategy of each buyer and each seller since all agents need to take into account the competition between the buyers. However, the increase of buyers will not change the choice rule at each information set and the number of choice trees. Therefore, we just need to adopt the analysis in Section 4 to find out agents' optimal strategies when there is competition between buyers.

Assume that at time $t + 1$ where $\iota(t + 1) = \mathcal{B}$, buyer types of $\mathbf{b}$ make the same offer $x^*_{\mathbf{b}}(t + 1)$, i.e., the pooling choice rule was chosen at time $t + 1$. Let the price of any other buyer $\mathbf{b}' \in \mathcal{B} - \mathbf{b}$ be $x^*_{\mathbf{b}'}(t + 1)$. When there is only one buyer, seller $\mathcal{S}^1_{t+2}$ will propose a price no lower than the offer of other sellers. Since seller $\mathcal{S}^1_{t+2}$ is uncertain of the type of the buyer, there are two options for seller $\mathcal{S}^1_{t+2}$. When there are multiple buyers, we can apply the result in Theorem 6 to find out $\mathcal{S}$'s highest equilibrium offer $x^*_{\mathcal{S}}(t)$ (or $x^*_{\mathbf{b}_h}(t)$) when $\mathbf{b}$ is of different types. Formally, when $\mathbf{b}$ is of type $\mathbf{b}_h$, $x^*_{\mathcal{S}}(t)$ is given by

$$\begin{cases} \{(x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h} \cup \{(x^*_{\mathbf{b}'}(t+1))_{\leftarrow \mathbf{b}'} | \mathbf{b}' \in \mathcal{B}_{t+3} - \mathbf{b}\} \cup \{\mathrm{RP}_{\mathbf{b}'} | \mathbf{b}' \in \mathcal{B}_{=t+2}\}\}^{|\mathcal{S}_{t+2}|} & \text{if } |\mathcal{S}_{t+2}| \leq |\mathcal{B}_{t+2}| \\ \min\{\mathrm{RP}_{\mathcal{S}^{|\mathcal{B}_{t+2}|+1}_{t+2}}, \{(x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}_h} \cup \{(x^*_{\mathbf{b}'}(t+1))_{\leftarrow \mathbf{b}'} | \mathbf{b}' \in \mathcal{B}_{t+3} - \mathbf{b}\} \cup \{\mathrm{RP}_{\mathbf{b}'} | \mathbf{b}' \in \mathcal{B}_{=t+2}\}\}^{|\mathcal{B}_{t+2}|}\} & \text{if } |\mathcal{S}_{t+2}| > |\mathcal{B}_{t+2}| \end{cases}$$

In the same way, we can define the value $x^*_{\mathcal{S}}(t)$ (or $x^*_{\mathbf{b}_l}(t)$) when $\mathbf{b}$ is of type $\mathbf{b}_l$. Each winning seller will choose one price from $x^*_{\mathbf{b}_h}(t)$ and $x^*_{\mathbf{b}_l}(t)$which can give it the highest expected utility. The equivalent price of each winning seller's offer can be computed based on the highest expected utility. If $\mathbf{b}$ chooses the separating rule at time $t + 1$, i.e., $\mathbf{b}_l$ makes an offer which will be rejected but $\mathbf{b}_h$ makes an acceptable offer. In this case, the value of $x^*_{\mathcal{S}}(t)$ can be computed in the same way as the buyer adopts the pooling choice rule except that the back propagated value $(x^*_{\mathbf{b}}(t + 1))_{\leftarrow \mathbf{b}_l}$ needs to be replaced by $(x^c_{\mathbf{b}_l}(t + 2))_{\leftarrow 2[\mathbf{b}_l]}$ since $x^*_{\mathbf{b}}(t + 1)$ will be rejected.

Now we analyze how to compute buyers' equilibrium offer at time t given sellers' equivalent offers at time $t + 1$. If the buyer adopts the pooling choice rule, we can apply the result in Theorem 6 to find out buyers' equilibrium offers. If the buyer adopts the separating choice rule, $\mathbf{b}_l$ will make an offer (say 0) that will be rejected and the offer of $\mathbf{b}_h$ will be accepted. Each buyer's optimal offer depends on the offer of other buyers. Thus, buyers will make proposals in Bayesian-Nash equilibrium at time t. Buyers' equilibrium strategies $\mathcal{X}^*_{\mathcal{B}}(t)$ at time t include the strategy of $\mathbf{b}_h$, $\mathbf{b}_l$, and other buyers. The utility of a buyer $\mathbf{b}' \in \mathcal{B}_{t+2} - \mathbf{b}$ is $\omega^0_{\mathbf{b}_h} U_{\mathbf{b}'}(\mathbf{b}_h) + \omega^0_{\mathbf{b}_l} U_{\mathbf{b}'}(\mathbf{b}_l)$ where $U_{\mathbf{b}'}(\mathbf{b}_h)$ ($U_{\mathbf{b}'}(\mathbf{b}_l)$) is the utility of $\mathbf{b}'$ when the buyer $\mathbf{b}$ is of type $\mathbf{b}_h$ ($\mathbf{b}_l$).

6 Analysis of Other Uncertainties

In this section we preliminarily analyze the impact of other types of uncertain information within our negotiation mechanism.

6.1 One-to-many setting

We initially focus on the computation of the equilibrium outcome with complete information. Although agents' equilibrium strategies depend on the values of the parameters of all the agents, for a large subset of the space of the parameters the equilibrium outcome depends on the values of a narrow number of parameters. We have the following theorem whose proof is in Appendix C.

Theorem 19. *When 1) $T_{\mathcal{S}_2^2} > 2$ if $\iota(0) = \mathbf{b}$ and 2) $(\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathcal{S}_2^1 \mathbf{b}} \geq \mathrm{RP}_{\mathbf{s}}$ for any seller $\mathbf{s} \in \mathcal{S}$, the equilibrium outcome depends only on the parameters of $\mathbf{b}$ (i.e., $\mathrm{RP}_{\mathbf{b}}$, $\delta_{\mathbf{b}}$, $T_{\mathbf{b}}$), $\mathcal{S}_2^1$ (i.e., $\mathrm{RP}_{\mathcal{S}_2^1}$, $\delta_{\mathcal{S}_2^1}$, $T_{\mathcal{S}_2^1}$), and on the reserve price $\mathrm{RP}_{\mathcal{S}_2^2}$ of $\mathcal{S}_2^2$. In these situations the equilibrium outcome can be produced as follows:*

1. *finding the sequence of the optimal offers under the assumption that $\mathcal{S}_2^1$ is the unique seller, say $y(t)$, and*
2. *assigning $x^*_{\mathbf{b}}(0) = \min\{y(0), (\mathrm{RP}_{\mathcal{S}_2^2})_{\leftarrow \mathcal{S}_2^1}\}$ if $\iota(0) = \mathbf{b}$ and assigning $x^*_{\mathcal{S}_2^1}(0) = \min\{y(0), \mathrm{RP}_{\mathcal{S}_2^2}\}$ if $\iota(0) = \mathcal{S}$.*

This is to say that the equilibrium outcome does not depend on the values of $\delta_{\mathcal{S}_2^2}$, $T_{\mathcal{S}_2^2}$, and on the parameters of all the other sellers. This is of paramount importance since complex settings with a high degree of uncertainty can be easily solved when 1) $T_{\mathcal{S}_2^2} > 2$ if $\iota(0) = \mathbf{b}$ and 2) $(\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathcal{S}_2^1 \mathbf{b}} \geq \mathrm{RP}_{\mathbf{s}}$ for any seller $\mathbf{s} \in \mathcal{S}$. Indeed, the above algorithm produces the equilibrium outcome even when $\delta_{\mathcal{S}_2^i}$ with $i > 1$, $T_{\mathcal{S}_2^i}$ with $i > 1$, and $\mathrm{RP}_{\mathcal{S}_2^i}$ with $i > 2$ are uncertain. We can write the condition $(\mathrm{RP}_{\mathcal{S}_2^2})_{\leftarrow \mathcal{S}_2^1 \mathbf{b}} \geq \mathrm{RP}_{\mathcal{S}_2^2}$ as

$$(\mathrm{RP}_{\mathbf{b}} - \mathrm{RP}_{\mathcal{S}_2^1}) \geq (\mathrm{RP}_{\mathcal{S}_2^2} - \mathrm{RP}_{\mathcal{S}_2^1}) \frac{1 - \delta_{\mathbf{b}} \delta_{\mathcal{S}_2^1}}{1 - \delta_{\mathbf{b}}}.$$

It can be easily observed that, in common real-world settings where $\mathrm{RP}_{\mathbf{b}} \gg \mathrm{RP}_{\mathcal{S}_2^2}$ and $\delta_{\mathcal{S}_2^1}$ is close to 1, the above condition is satisfied.

Now, we focus on the uncertainty over $\mathbf{b}$'s and $\mathcal{S}_2^1$'s parameters. The values of these parameters affect the equilibrium outcome and therefore in presence of uncertainty over them we need to compute agents' equilibrium strategies to derive the equilibrium outcome. Currently, the literature provides algorithms to compute agents' equilibrium strategies only in bilateral settings without outside option with one-sided uncertainty over deadlines [11]. We recall that, since the number of available actions is infinite, no algorithms such as Lemke-Howson [29] can be employed to compute a sequential equilibrium.

When $\mathrm{RP}_{\mathcal{S}_2^2} \leq (\mathrm{RP}_{\mathcal{S}_2^2})_{\leftarrow \mathcal{S}_2^1 \mathbf{b}}$, the algorithm presented in [11] can be easily extended to capture uncertainty in one-to-many bargaining. More precisely, we have that:

- when $T_{\mathbf{b}}$ is uncertain, whereas $T_{\mathcal{S}_2^1}$ is certain, then agents' equilibrium strategies can be produced by employing the algorithm presented in [11] where the buyer is $\mathbf{b}$ and the seller is $\mathcal{S}_2^1$ and upper bounding the optimal offers to $\mathrm{RP}_{\mathcal{S}_2^2}$ if $\iota(0) = \mathbf{b}$ and to $(\mathrm{RP}_{\mathcal{S}_2^2})_{\leftarrow \mathcal{S}_2^1}$ if $\iota(t) = \mathcal{S}$;
- when $T_{\mathcal{S}_2^1}$ is uncertain, whereas $T_{\mathbf{b}}$ is certain, then agents' equilibrium strategies can be computed.

Settings with a higher degree of uncertainty, such as when both $T_{\mathbf{b}}$ and $T_{\mathcal{S}_2^1}$ are uncertain, need further exploration.

The results discussed above show that the analytical complexity of one-to-many bargaining is drastically less complicated than that of bilateral bargaining with outside option. This allows one to drastically reduce the search space and makes the computation easy. Therefore, one-to-many bargaining seems more appropriate for real-world settings when computational issues should be considered.

6.2 Many-to-many setting

In this section we provide some considerations on the preliminary analysis of many-to-many bargaining with uncertainty over agents' parameters. The result discussed in Section 6.1 can be treated as a special case for many-to-many bargaining. With more buyers, the agreement price will increase due to the increasing competition between buyers. For the bargaining between buyers $\mathcal{B}$ and sellers $\mathcal{S}$, it can be found from Theorem 6 that the agreement price depends on the reserve price of at least $\min\{|\mathcal{B}|, |\mathcal{S}|\}$ buyers and $\min\{|\mathcal{B}|, |\mathcal{S}|\}$ sellers. Although the many-to-many bargaining setting is intrinsically very complicated, the problem of finding the equilibrium outcome can be drastically simplified in some special cases.

Theorem 20. *In the following many-to-many bargaining scenarios in which $|\mathcal{B}| < |\mathcal{S}|$, the negotiation outcome only depends on the parameters of $\mathcal{B}$ and at most $|\mathcal{B}| + 1$ sellers:*

1. *The sellers having a reserve price no higher than the $\mathrm{RP}_{\mathcal{S}_2^{|\mathcal{B}|+1}}$ have the same deadline T' such that $\iota(T') = \mathcal{S}$.*
2. *At each time t, the seller set $\mathcal{S}_{t+2}$ includes all the sellers with a reserve price no higher than $\mathrm{RP}_{\mathcal{S}_2^{|\mathcal{B}_{t+2}|+1}}$.*

Proof. Case 1: At time $T' - 2$, the value of $x^*_{\mathcal{S}}(T' - 2)$ should be no higher than $\mathrm{RP}_{\mathcal{S}_2^{|\mathcal{B}|+1}}$ and is independent of the reserve prices of sellers having a reserve price higher than $\mathrm{RP}_{\mathcal{S}_2^{|\mathcal{B}|+1}}$. At time $t = T' - 3$, the value of $x^*_{\mathcal{B}}(t)$ will also be no higher than $\mathrm{RP}_{\mathcal{S}_2^{|\mathcal{B}|+1}}$. Recursively, we can find that the the value of $x^*_{\mathcal{B}}(t)$ at time $t < T' - 3$ will be no higher than $\mathrm{RP}_{\mathcal{S}_2^{|\mathcal{B}|+1}}$ and is independent of the reserve prices of sellers having a reserve price higher than $\mathrm{RP}_{\mathcal{S}_2^{|\mathcal{B}|+1}}$.

Case 2: We can prove the result in the same way as in the proof of *Case 1*. □

Thus, the negotiation outcome only depends on a small number of parameters in some special cases. The complexity of solving complete information bargaining and incomplete information bargaining can be reduced.

7 Conclusion

This paper analyzes agents' strategic behavior in concurrent one-to-many negotiation and many-to-many negotiation when agents follow the alternating-offers protocol. For complete information settings, we show that the computational complexity when there are many buyers and many sellers in our protocol is essentially the same in the situation where the negotiations are purely bilateral. We also preliminarily explore how uncertainty over reserve prices and deadlines can affect equilibrium strategies. We observe that agents' bargaining power are affected by the proposing ordering and market competition. We also compare the efficiency of the negotiation mechanism with that of some other mechanisms like VCG auction. We also find that the computation of the equilibrium for realistic ranges of the parameters in one-to-many settings reduces to the computation of the equilibrium either in one-to-one settings with uncertainty or in one-to-many settings without uncertainty.

One major motivation of the study of bargaining theory is designing successful bargaining agents in practical dynamic markets where agents often have to negotiate with multiple trading partners while facing the competition from agents of the same type. Although constraints, complexity, and uncertainty make it impractical to develop optimal negotiation strategies, our analysis can still give us some insights into the bargaining problems. This paper provides some useful guidelines for designing negotiation agents. For example, market competition plays a central role in deciding the market equilibrium, agents need to make the same offer to all the trading partners at each time.

Another future research direction is theoretically analyzing agents' strategic behavior in one-to-many negotiation and many-to-many negotiation when agents have incomplete information about more than one agent's reserve prices, and discounting factors. Moreover, another level of uncertainty which comes with one-to-many and many-to-many negotiation is that an agent may only have probabilistic information about the number of trading parters and trading competitors. When each trader privately knows its own preferences, it may have an incentive to misrepresent its preference in order to influence the market equilibrium in its favor and it will learn the other agents' preferences during the bargaining process. A number of bargaining models [9, 11, 20, 23] have studied incomplete information bargaining and some surprising results show that learning won't happen in some situations (e.g., [20]). It would be interesting to investigate an agent's incentive to misrepresent its preference in a market where a single agent's influence on the market equilibrium will decrease with the increase of the scale of the market.

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A Proof of Theorem 2

Proof. First compute agents' optimal offers using backward induction. Let $x^*_{\mathcal{S}}(t) = \min_{\mathbf{s}_i \in \mathcal{S}_{t+2}} x^*_{\mathbf{s}_i}(t)$ be $\mathcal{S}$'s highest optimal offer at t. It follows that $x^*_{\mathbf{s}_i}(t) = \max\{\mathrm{RP}_{\mathbf{s}_i}, x^*_{\mathcal{S}}(t)\}$. At time point T, the game for the buyer $\mathbf{b}$ rationally stops. The equilibrium outcome of every subgame starting from $t \geq T$ is $NoAgreement$. Therefore, at $t = T$ agent $\iota_{\Im_{\mathbf{b},\mathbf{s}_i}}(T)$ would only confirm the best agreement proposed by agent $\iota_{\Im_{\mathbf{b},\mathbf{s}_i}}(T-1)$. At time $t = T-1$, $\iota_{\Im_{\mathbf{b},\mathbf{s}_i}}(T-1)$ will accept the best offer by agent $\iota_{\Im_{\mathbf{b},\mathbf{s}_i}}(T-2)$, if $\iota_{\Im_{\mathbf{b},\mathbf{s}_i}}(T-1)$ can get a utility not worse than $NoAgreement$ by accepting the best offer. Note that at time $T-1$ and T, no agent will propose a price as it takes at least three time points to implement a final contract.

Assume that $\iota_{\Im_{\mathbf{b},\mathbf{s}_i}}(t) = \mathbf{b}$. If $t = T-2$ or $t = T_{\mathcal{S}^1_{t+2}} - 2$, $\mathbf{b}$'s optimal price is $\mathrm{RP}_{\mathcal{S}^1_{t+2}}$ and seller $\mathcal{S}^1_{t+2}$ will accept it as its deadline is approaching. At $t < T-2$, $\min_{\mathbf{s}_i \in \mathcal{S}_{t+3}} \big((x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i}\big)$ is surely acceptable to some sellers in $\mathcal{S}_{t+3}$. We also need to consider sellers $\mathcal{S}_{t+2} - \mathcal{S}_{t+3}$ with deadline $t+2$, who are willing to accept any offer which is no less than their reserve prices. Therefore, $\mathbf{b}$'s optimal offer at time t is

$$x^*_{\mathbf{b}}(t){=}\min\{\min_{\mathbf{s}_i \in \mathcal{S}_{t+3}} \big((x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i}\big), \min_{\mathbf{s}_i \in \mathcal{S}_{t+2} - \mathcal{S}_{t+3}} \mathrm{RP}_{\mathbf{s}_i}\} \tag{1}$$

It is easy to see that $x^*_{\mathcal{S}^1_{t+3}}(t+1) \leq x^*_{\mathcal{S}^2_{t+3}}(t+1) = \mathrm{RP}^*_{\mathcal{S}^2_{t+3}}(t+1)$. It follows that $\min_{\mathbf{s}_i \in \mathcal{S}_{t+3}} \big((x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i}\big) = (x^*_{\mathcal{S}^1_{t+3}}(t+1))_{\leftarrow \mathcal{S}^1_{t+3}}$. As $t \neq T_{\mathcal{S}^1_{t+2}} - 2$, equation (1) can be rewritten as $\min\{(x^*_{\mathcal{S}^1_{t+2}}(t+1))_{\leftarrow \mathcal{S}^1_{t+2}}, \mathrm{RP}_{\mathcal{S}^2_{t+2}}\}$. Therefore, $x^*_{\mathbf{b}}(t) = \min\{(x^*_{\mathcal{S}^1_{t+2}}(t+1))_{\leftarrow \mathcal{S}^1_{t+2}}, \mathrm{RP}_{\mathcal{S}^2_{t+2}}\}$ if $t < T-2$ and $t \neq T_{\mathcal{S}^1_{t+2}} - 2$.

Assume that $\iota_{\Im_{\mathbf{b},\mathbf{s}_i}}(t) = \mathbf{s}_i$. At time $t = T-2$, the acceptable offer to buyer $\mathbf{b}$ is $\mathrm{RP}_{\mathcal{S}^2_T}$ as all sellers in $\mathcal{S}^2_T$ compete with each other to get a contract. Thus, $\mathbf{s}_i$'s optimal offer is $\max\{\mathrm{RP}_{\mathbf{s}_i}, \mathrm{RP}_{\mathcal{S}^2_T}\}$. At time $t < T-2$, the acceptable offer to buyer $\mathbf{b}$ is $(x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}}$. However, $\mathbf{s}_i$ needs to consider the competition among sellers then $\mathbf{s}_i$'s winning price should be no higher than $\mathrm{RP}_{\mathcal{S}^2_{t+2}}$. Then $\mathbf{s}_i$'s optimal offer is $\max\{\mathrm{RP}_{\mathbf{s}_i}, \min\{\mathrm{RP}_{\mathcal{S}^2_{t+2}}, (x^*_{\mathbf{b}}(t+1))_{\leftarrow \mathbf{b}}\}\}$.

Finally, agents' optimal actions can be easily defined on the basis of $x^*_{\mathbf{b}}(t)$ and $x^*_{\mathbf{s}_i}(t)$. When an agent decides to make an offer, it always proposes it optimal offer ($x^*_{\mathbf{b}}(t)$ or $x^*_{\mathbf{s}_i}(t)$). Buyer $\mathbf{b}$ will accept an offer $\sigma_{\mathbf{s}_i}(t-1)$ if $\sigma_{\mathbf{s}_i}(t-1) \leq (x^*_{\mathbf{b}}(t))_{\leftarrow \mathbf{b}}$ and $\sigma_{\mathbf{s}_i}(t-1)$ is no higher than other sellers' offers at time $t-1$. It is possible that several sellers propose a same acceptable offer. The tie can be broken by choosing the lowest offer from the seller with the lowest reserve price (note that we assume that sellers have different reserve prices). If at time $t-1$, seller $\mathbf{s}_i$ agrees with $\mathbf{b}$'s offer $\sigma_{\mathbf{b},\mathbf{s}_i}(t-2)$, $\mathbf{b}$ will confirm the agreement if $\sigma_{\mathbf{b},\mathbf{s}_i}(t-2) \leq \sigma_{\mathbf{b},\mathbf{s}_j}(t-2)$ if $\mathbf{s}_j$ also agrees with $\mathbf{b}$'s offer at time $t-1$. Again, there could be more than one agreement with the same lowest price. To make sure that $\mathbf{b}$ only makes one final agreement, $\mathbf{b}$ confirms the agreement from the seller with the lowest reserve price. The optimal actions of all the sellers can be defined analogously. For simplicity, we consider just agents' strategies on the equilibrium path. □

B Proof of Theorem 6

Proof. Given Lemma 4 and Lemma 5, we just need to find out the agents' equilibrium winning price at each time point. Let $x^*_{\mathcal{B}}(t)$ ($x^*_{\mathcal{S}}(t)$) be $\mathcal{B}$'s lowest ($\mathcal{S}$'s highest) offer which is acceptable to $\mathcal{S}$ ($\mathcal{B}$) at time t if $\iota(t) = \mathcal{B}$ ($\iota(t) = \mathcal{S}$). It follows that $x^*_{\mathcal{B}}(t) = \max_{\mathbf{b}_j \in \mathcal{B}_{t+2}} x^*_{\mathbf{b}_j}(t)$, $x^*_{\mathcal{S}}(t) = \min_{\mathbf{s}_i \in \mathcal{S}_{t+2}} x^*_{\mathbf{s}_i}(t)$.

Following the idea of backward induction, at $T = \max_{\mathbf{b}_j \in \mathcal{B}} T_{\mathbf{b}_j, \mathcal{S}}$, the game for all agents rationally stops. The equilibrium outcome of every subgame starting from $t \geq T$ is $NoAgreement$. Therefore, at $t = T$, agents $\iota(T)$ would only confirm the best agreement proposed by agents $\iota(T-1)$. At time $t = T-1$, agents $\iota(T-1)$ will accept the best offer by agents $\iota(T-2)$ if the best offer is no worse than $NoAgreement$ by accepting the best offer. At time $T-1$ and T, no agent will propose a price as it takes at least three time points to implement a final contract.

At time $t = T - 2$, agents $\iota(t)$ will strive to make the best offer. There are two situations: $\iota(t) = \mathcal{B}$ or $\iota(t) = \mathcal{S}$. First consider the case $\iota(t) = \mathcal{B}$ and there are two cases: Case 1 ($|\mathcal{B}_T| \leq |\mathcal{S}_T|$): In this case, the supply is no less than demand and buyers have more bargaining power as compared with sellers. It is easy to see that each buyer's optimal price is $\mathrm{RP}_{\mathcal{S}_T^{|\mathcal{B}_T|}}$ as, by doing so, $|\mathcal{B}_T|$ sellers will agree to sell their good and each buyer can get a good. If one buyer pays less than $\mathrm{RP}_{\mathcal{S}_T^{|\mathcal{B}_T|}}$, the sellers will choose another buyers paying $\mathrm{RP}_{\mathcal{S}_T^{|\mathcal{B}_T|}}$. It doesn't make sense that a rational agent will pay more than $\mathrm{RP}_{\mathcal{S}_T^{|\mathcal{B}_T|}}$. If each buyer pays a price less than $\mathrm{RP}_{\mathcal{S}_T^{|\mathcal{B}_T|}}$, each buyer will face a risk of losing an agreement as the number of sellers who are willing to accept the price is less than the number of buyers. Case 2 ($|\mathcal{B}_T| > |\mathcal{S}_T|$): In this case, the supply is less than demand and buyers need to compete with each other to get agreements. It is easy to say that each buyer's optimal price is $\mathrm{RP}_{\mathcal{B}_T^{|\mathcal{S}_T|+1}}$. In the same way, we can get the optimal offer of buyers $\mathcal{S}_T$ at time $T-2$: $x^*_{\mathcal{S}}(T-2) = \mathrm{RP}_{\mathcal{B}_T^{|\mathcal{S}_T|}}$ if $|\mathcal{S}_T| \leq |\mathcal{B}_T|$, $x^*_{\mathcal{S}}(T-2) = \mathrm{RP}_{\mathcal{S}_T^{|\mathcal{B}_T|+1}}$ if $|\mathcal{S}_T| > |\mathcal{B}_T|$.

Then we move to the calculation for computing $x^*_{\mathcal{B}}(t)$ and $x^*_{\mathcal{S}}(t)$ given $x^*_{\mathcal{B}}(t+1)$ and $x^*_{\mathcal{S}}(t+1)$. First consider the situation that $\iota(t) = \mathcal{B}$. There are two situations depending on whether there are agents with deadline $t+2$. If there is no agent with deadline $t+2$, $(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i}$ is surely acceptable to seller $\mathbf{s}_i$ at time t. Here we consider two cases: 1) $|\mathcal{S}_{t+3}| \geq |\mathcal{B}_{t+3}|$. It is easy to see that, the price $\min_{\mathbf{s}_i \in \mathcal{S}_{t+3}} \big((x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i}\big)$ is surely acceptable to sellers in $\mathcal{S}_{t+3}$ whose optimal price is $x^*_{\mathcal{S}}(t+1)$ at time $t+1$. However, we also need to consider the competition among buyers. Therefore, $x^*_{\mathcal{B}}(t) = \{(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+3}\}_{|\mathcal{B}_{t+3}|}$ where $\mathcal{Y}_i$ ($\mathcal{Y}^i$) is the i^{th} smallest (largest) value in the value set $\mathcal{Y}$. 2) $|\mathcal{S}_{t+3}| < |\mathcal{B}_{t+3}|$. As $(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} \leq x^*_{\mathbf{s}_i}(t+1)$, $x^*_{\mathcal{B}}(t)$ should be no less than $\mathrm{RP}_{\mathcal{B}_{t+3}^{|\mathcal{S}_{t+3}|+1}}$. Therefore, it follows that $x^*_{\mathcal{B}}(t) = \max\big\{\mathrm{RP}_{\mathcal{B}_{t+3}^{|\mathcal{S}_{t+3}|+1}}, \{(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+3}\}_{|\mathcal{S}_{t+3}|}\big\}$.

Now we move to the general case that there are some buyers or sellers with deadline $t+2$. For a buyer with deadline $t+2$, it is willing to propose its reserve price. For a seller with deadline $t+2$, it is willing to accept an offer of its reserve price. Assume that there are only some sellers with deadline $t+2$. We consider three cases: 1) $|\mathcal{S}_{t+3}| \geq |\mathcal{B}_{t+3}|$, which implies that $|\mathcal{S}_{t+2}| > |\mathcal{B}_{t+2}|$. It is easy to see that, $x^*_{\mathcal{B}}(t) = \big\{\{(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+2} - \mathcal{S}_{t+3}\}\big\}_{|\mathcal{B}_{t+2}|}$. 2) $|\mathcal{S}_{t+3}| < |\mathcal{B}_{t+3}|$ and $|\mathcal{S}_{t+2}| < |\mathcal{B}_{t+2}|$. In this case, $x^*_{\mathcal{B}}(t) = \max\big\{\mathrm{RP}_{\mathcal{B}_{t+2}^{|\mathcal{S}_{t+2}|+1}}, \big\{\{(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+2} - \mathcal{S}_{t+3}\}\big\}_{|\mathcal{S}_{t+2}|}\big\}$. 3) $|\mathcal{S}_{t+3}| < |\mathcal{B}_{t+3}|$ and $|\mathcal{S}_{t+2}| \geq |\mathcal{B}_{t+2}|$. In this case, $x^*_{\mathcal{B}}(t) = \big\{\{(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+2} - \mathcal{S}_{t+3}\}\big\}_{|\mathcal{B}_{t+2}|}$.

We can easily extend the above analysis to more general cases where there are both buyers and sellers with deadline $t+2$. We can get $\mathcal{B}$'s optimal price at time $t < T-2$ as follows: 1) if $|\mathcal{S}_{t+2}| < |\mathcal{B}_{t+2}|$, $x^*_{\mathcal{B}}(t) = \max\big\{\mathrm{RP}_{\mathcal{B}_{t+2}^{|\mathcal{S}_{t+2}|+1}}, \big\{\{(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+2} - \mathcal{S}_{t+3}\}\big\}_{|\mathcal{S}_{t+2}|}\big\}$; 2) otherwise, $x^*_{\mathcal{B}}(t) = \big\{\{(x^*_{\mathbf{s}_i}(t+1))_{\leftarrow \mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{s}_i} | \mathbf{s}_i \in \mathcal{S}_{t+2} - \mathcal{S}_{t+3}\}\big\}_{|\mathcal{B}_{t+2}|}$.

In the same way, we can get $\mathcal{S}$'s optimal price at time $t < T-2$ as follows: 1) if $|\mathcal{S}_{t+2}| \leq |\mathcal{B}_{t+2}|$, $x^*_{\mathcal{S}}(t) = \big\{\{x^*_{\mathbf{b}_j}(t+1)_{\leftarrow \mathbf{b}_j} | \mathbf{b}_j \in \mathcal{B}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{b}_j} | \mathbf{b}_j \in \mathcal{B}_{=t+2}\}\big\}^{|\mathcal{S}_{t+2}|}$; 2) otherwise, $x^*_{\mathcal{S}}(t) = \min\big\{\mathrm{RP}_{\mathcal{S}_{t+2}^{|\mathcal{B}_{t+2}|+1}}, \big\{\{(x^*_{\mathbf{b}_j}(t+1))_{\leftarrow \mathbf{b}_j} | \mathbf{b}_j \in \mathcal{B}_{t+3}\} \cup \{\mathrm{RP}_{\mathbf{b}_j} | \mathbf{b}_j \in \mathcal{B}_{=t+2}\}\big\}^{|\mathcal{B}_{t+2}|}\big\}$. □

C Proof of Theorem 19

Proof. Case 1 ($\iota(\min\{T_{\mathcal{S}_2^1}, T_{\mathbf{b}}\}) = \mathbf{b}$). Let $t'+2 = \min\{T_{\mathcal{S}_2^1}, T_{\mathbf{b}}\}$. It's easy to see that $x^*_{\mathbf{b}}(t') = \mathrm{RP}_{\mathcal{S}_2^1} = y(t')$. Then we have $x^*_{\mathcal{S}}(t'-1) = \min\{(\mathrm{RP}_{\mathcal{S}_2^1})_{\leftarrow b}, \mathrm{RP}_{\mathcal{S}^2_{t'+1}}\} = \min\{y(t'-1), \mathrm{RP}_{\mathcal{S}^2_{t'+1}}\}$.[2] At time $t'-2$, we have $x^*_{\mathbf{b}}(t'-2) = \min\{(\mathrm{RP}_{\mathcal{S}_2^1})_{\leftarrow b \mathcal{S}_2^1}, (\mathrm{RP}_{\mathcal{S}^2_{t'+1}})_{\leftarrow \mathcal{S}_2^1}, \mathrm{RP}_{\mathcal{S}^1_{=t'}}\} = \min\{y(t'-2), (\mathrm{RP}_{\mathcal{S}^2_{t'+1}})_{\leftarrow \mathcal{S}_2^1}, \mathrm{RP}_{\mathcal{S}^1_{=t'}}\}$.

[2] For convenience, $\mathrm{RP}_{\mathcal{S}^2_{t'+1}} = \infty$ if $|\mathcal{S}_{t'+1}| < 2$.

At time $t'-3$, we have $x^*_{\mathcal{S}}(t'-3)=\min\{(y(t'-2))_{\leftarrow\mathbf{b}},(\mathrm{RP}_{\mathcal{S}^2_{t'+1}})_{\leftarrow\mathcal{S}^1_2\mathbf{b}},(\mathrm{RP}_{\mathcal{S}^1_{=t'}})_{\leftarrow\mathbf{b}},\mathrm{RP}_{\mathcal{S}^2_{t'-1}}\}$. It's obvious that $(\mathrm{RP}_{\mathcal{S}^1_{=t'}})_{\leftarrow\mathbf{b}}\geq(\mathrm{RP}_{\mathcal{S}^1_{=t'}})\geq\mathrm{RP}_{\mathcal{S}^2_{t'-1}}$. In addition, as we assume that $(\mathrm{RP}_{\mathbf{s}})_{\leftarrow\mathcal{S}^1_2\mathbf{b}}\geq\mathrm{RP}_{\mathbf{s}}$, it follows that $(\mathrm{RP}_{\mathcal{S}^2_{t'+1}})_{\leftarrow\mathcal{S}^1_2\mathbf{b}}\geq\mathrm{RP}_{\mathcal{S}^2_{t'+1}}\geq\mathrm{RP}_{\mathcal{S}^2_{t'-1}}$. Then we have $x^*_{\mathcal{S}}(t'-3)=\min\{y(t'-3),\mathrm{RP}_{\mathcal{S}^2_{t'-1}}\}$. Following this procedure, we have 1) if $\iota(0)=\mathcal{S}$, $x^*_{\mathcal{S}^1_2}(0)=\min\{y(0),\mathrm{RP}_{\mathcal{S}^2_2}\}$; 2) if $\iota(0)=\mathbf{b}$, $x^*_{\mathbf{b}}(0)=\min\{y(0),(\mathrm{RP}_{\mathcal{S}^2_2})_{\leftarrow\mathcal{S}^1_2}\}$ as $(\mathrm{RP}_{\mathcal{S}^2_3})_{\leftarrow\mathcal{S}^1_2}=(\mathrm{RP}_{\mathcal{S}^2_2})_{\leftarrow\mathcal{S}^1_2}\leq(\mathrm{RP}_{\mathcal{S}^2_2})\leq\mathrm{RP}_{\mathcal{S}^3_2}\leq\mathrm{RP}_{\mathcal{S}^1_{=2}}$ given that $T_{\mathcal{S}^2_2}>2$.
Case 2 ($\iota(\min\{T_{\mathcal{S}^1_2},T_{\mathbf{b}}\})=\mathcal{S}$). Let $t'+2=\min\{T_{\mathcal{S}^1_2},T_{\mathbf{b}}\}$. At time t', there are two situations: 1) $|\mathcal{S}_{t'+2}|<2$, which implies that $x^*_{\mathcal{S}}(t')=\mathrm{RP}_{\mathbf{b}}=y(t')$; 2) Otherwise, $x^*_{\mathcal{S}}(t')=\mathrm{RP}_{\mathcal{S}^2_{t'+2}}$. Therefore, $x^*_{\mathcal{S}}(t')=\min\{y(t'),\mathrm{RP}_{\mathcal{S}^2_{t'+2}}\}$. At time $t'-1$, it follows that $x^*_{\mathbf{b}}(t'-1)=\min\{y(t'-1),(\mathrm{RP}_{\mathcal{S}^2_{t'+2}})_{\leftarrow\mathcal{S}^1_2},\mathrm{RP}_{\mathcal{S}^1_{=t'+1}}\}$. Then at time $t'-2$, we have $x^*_{\mathcal{S}}(t'-2)=\min\{y(t'-2),(\mathrm{RP}_{\mathcal{S}^2_{t'+2}})_{\leftarrow\mathcal{S}^1_2\mathbf{b}},(\mathrm{RP}_{\mathcal{S}^1_{=t'+1}})_{\leftarrow\mathbf{b}},\mathrm{RP}_{\mathcal{S}^1_{=t'}},\mathrm{RP}_{\mathcal{S}^2_{t'}}\}$. It easy to see that $\mathrm{RP}_{\mathcal{S}^1_{=t'}}\geq\mathrm{RP}_{\mathcal{S}^2_{t'}}$. It's obvious that $(\mathrm{RP}_{\mathcal{S}^1_{=t'+1}})_{\leftarrow\mathbf{b}}\geq(\mathrm{RP}_{\mathcal{S}^1_{=t'+1}})\geq\mathrm{RP}_{\mathcal{S}^2_{t'}}$. As we assume that $(\mathrm{RP}_{\mathbf{s}})_{\leftarrow\mathcal{S}^1_2\mathbf{b}}\geq\mathrm{RP}_{\mathbf{s}}$, it follows that $(\mathrm{RP}_{\mathcal{S}^2_{t'+2}})_{\leftarrow\mathcal{S}^1_2\mathbf{b}}\geq\mathrm{RP}_{\mathcal{S}^2_{t'+2}}\geq\mathrm{RP}_{\mathcal{S}^2_{t'}}$. Then we have $x^*_{\mathcal{S}}(t'-2)=\min\{y(t'-2),\mathrm{RP}_{\mathcal{S}^2_{t'}}\}$. Following this procedure, we have 1) if $\iota(0)=\mathcal{S}$, $x^*_{\mathcal{S}^1_2}(0)=\min\{y(0),\mathrm{RP}_{\mathcal{S}^2_2}\}$; 2) if $\iota(0)=\mathbf{b}$, $x^*_{\mathbf{b}}(0)=\min\{y(0),(\mathrm{RP}_{\mathcal{S}^2_2})_{\leftarrow\mathcal{S}^1_2}\}$ given that $T_{\mathcal{S}^2_2}>2$. □